Convex analysis and mathematical economics

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Research memorandum

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Research Memorandum

v convex functions

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Foreword.

On February 20, 1978, the Department of Econometrics of the University of Tilburg organized a symposium on Convex Analysis and Mathematical Economics, to commemorate the 50^{th} anniversary of the University. The general theme of the anniversary celebration was "innovation" and since an important part of the departments' theoretical work is concentrated on mathematical economics, the above mentioned theme was chosen. The scientific part of the Symposium consisted of four lectures:

The contents of these lectures are collected in this research memorandum.

J. Kriens

Chairman Organizing Committee.

ON THE DEVELOPMENT OF THE APPLICATION OF CONVEXITY

L.R.J. Westermann

This is not a survey. It aims, by a description of a few topics, at pointing out that, in the area of convexity, problems are rather stable but methods more or less perturb. For references with a less superficial treatment one can consult for section 1: [8], [12]; for section 2: [3], [5], [7], [11], Bnd II; for section 3: [1], [4], [9], [10], [14].

Already traditional school-geometry is full of convexity without an explicit mentioning of it. The strong attention there for the triangle seems to stress the importance of convexity since any triangle is convex and because all kinds of geometrical configurations can be pieced together from triangles. Triangulations and triangular approximation did show up in Euclid's Elements and their role is still dominant in to-day mathematics. Anyhow, this view is in line with the wellknown tendency of those who study the history of mathematics to trace theories back to ancient times and Euclid. From those times too stems the isoperimetric problem: "to find the plane figure of given perimeter with largest enclosed area", of which the solution is historically also tied up with convexity. Although in the history of science the notion of convexity plays a role, its part in the acting was rather hidden until the second part of last century.

From that time on it is on the scene of mathematics and its applications and has penetrated in several directions. Problems in number theory and related geometry directed explicit attention to convex sets. These first investigations led quickly to an appealing, geometrically simple, but powerfull machinery of notions and properties, such as gauge-functions, support-functions, separation, extremal sets, polars, mixed-volumes, symmetrization and so on. Besides that, the rise of convexity suited well in some developments since then. The axiomatized theory of locally convex topological spaces as a generalized outgrowth of finite dimensional vector-spaces, Hilbert- and Banach-spaces demonstrates the crucial place of convexity. The still broadening attention for optimization-theory and -methods gave ample room to convexity with it specific answers to questions of existence and uniqueness; in this area convexity helped also to link traditional optimization theory by duality with the expanded needs of present-day science, particularly mathematical economics. In differential-geometry the popular theme "from local to global" got a specific approach from the point of view of convex surfaces.

This lecture is now divided in three parts: 1. some considerations on the convexity concept; 2. a schematic treatment of the isoperimetric problem that might demonstrate classical but not at all obsolescent convexity-means in operation; 3. some remarks on convex surfaces, thus continuing the line.

Since our aim is to illustrate some elementary geometrical ideas involved, we shall try to avoid technicalities and shall feel free in the use of pictures of almost misleading simplicity.

1. ON CONVEXITY

A set C of some space E is said to be convex iff with each pair $p,q \in C$ $(,p \neq q,)$ at least one element between p and q belongs to C ,cf. [2] ; therefore a consistent notion of betweennes is needed. This can be deduced from the metric if E is a metric space or it can be generated by a real linear structure on E. Both situations occur. E can be taken a (complete, arcwise connected) curved surface in space and the distance $d(p,q)$ as the infimum of the set of lenghts of curves on E joining p and q; now r is between p and q iff $d(p,r) + d(r,q) = d(p,q)$, (Fig, 1-a).
For the second case we take E to be a real linear space and then $r \in E$ is between $p,q \in E$ iff $r = \alpha p + (1-\alpha)q$ for some $\alpha \in (0,1)$, (Fig. 1-b).

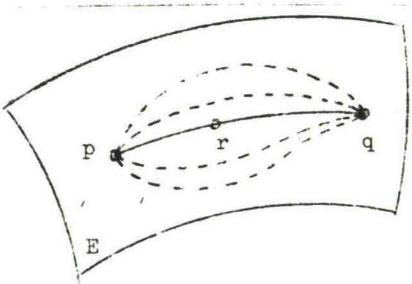

Fig. 1-a

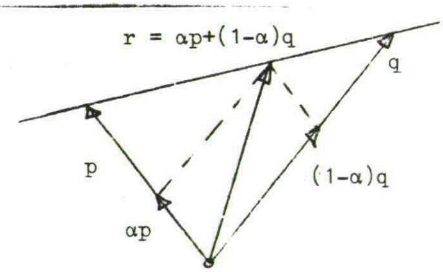

Fig. 1-b

Either way gives back Minkowski's original definition for a closed set C of euclidean space E^n: "C is convex iff it contains for each pair of its points the segment joining them". This definition can however equally well be applied for arbitrary sets in an arbitrary real linear space (, even a complex space). The transition from finite to infinite dimensional spaces is illuminating for convexity's role in the interplay between the algebraic and the topological structure. It is well known that in E^n each point p can be separated strictly from a closed convex set C not containing p, i.e. there exists a linear function f on E^n such that $f(p) < \inf \{ f(c) | c \in C \}$ (Fig. 2). For infinite dimensional spaces the topological structure becomes discriminating. The foregoing brings us to the following approach to the basic notion of convexity.

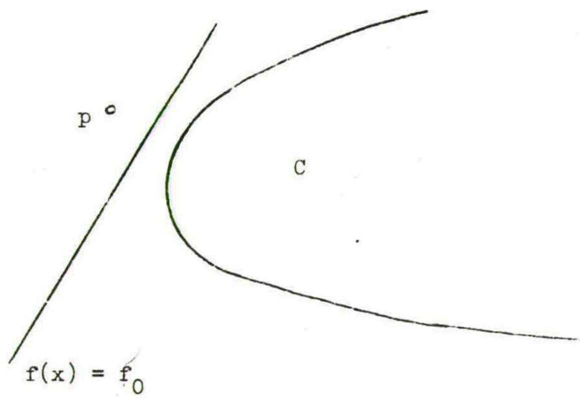

Fig. 2. $f(p) < f_0 < f(c)$, $\forall c \in C$.

Let E be a real linear space, E^* its algebraic dual (, i.e. the linear space of linear functions on E) and K some subspace of E^*. Consider the class

$$C_K := \{ C \subset E \mid \forall x \in E \backslash C \ \exists k \in K : k(x) < \inf \{k(c) \mid c \in C\}\}$$

and furnish E with the topology T_K determined by C_K as a subbase for the closed sets (; the closed sets of E are then intersections of finite unions of members of C_K). For any choice for K the linear space E will become a topological vectorspace with K as its topological dual. Next call simply a subset of E K-convex iff it belongs to C_K.
These K-convex sets are convex sets in

the ordinary sense and they form a closed family with respect to arbitrary intersections. Once the topology is determined the class of convex sets can be enlarged with open convex sets; these can be defined as open subsets U of E such that for each $x \in E\backslash U$ there exists a $k \in K$ with $k(x) \leqq \inf \{k(u) \mid u \in U\}$. Such a set U is then convex in the ordinary sense and besides its closure $\bar{U} \in C_K$. In this way with the separation-concept as a fundament the topology of E is defined and simultaneously the more interesting convex subsets of E are characterized, while pathological convex sets, as for instance convex dense proper subsets of E, do not even enter the theory. The class of convex sets depends on the chosen dual, but for a fine enough structure we want, as usual, points of E to belong to the class C_K of closed convex sets and this Hausdorff-requirement demands K to contain for each non-null $x \in E$ a k with $k(x) \neq 0$. In this way E gets a neighbourhood-base $\mathcal{U}$ of open convex sets and becomes therefore a locally convex topological space.

Apart from the way convexity is introduced let us consider a real locally convex topological linear space E. Then for each $U \in \mathcal{U}$ the function $p : E \to \mathbb{R}_+$, defined by

$$p(x) := \inf \{\lambda \mid \lambda > 0,\ x \in \lambda U\}$$

, called the Minkowski-functional or gauge of U, has the properties

$$p(x) \geqq 0 \ , \ p(\alpha x) = |\alpha| \cdot p(x) \ , \ p(x+y) \leqq p(x) + p(y) \ ,$$

for all $x,y \in E$ and all $\alpha \in \mathbb{R}$. This means that p is a semi-norm. The set $B \subset E$ is called bounded iff it is absorbed by every neighbourhood of 0; this is guaranteed if to each $U \in \mathcal{U}$ there is a $\lambda > 0$ with $B \subset \lambda U$. If now E is such that 0 has a bounded neighbourhood V then E is even a normed space and a $U \in \mathcal{U}$, which is contained in V, has as Minkowski-functional a norm for E; a norm determined by a set as its indicatrix, as for instance the euclidean norm by the unit ball, is a well-known concept from geometry, [11].

There is some temptation to dwell a little more on these abstract matters and discuss the fundamental significance of several other simple convexity-notions, as e.g. polars, support-functions and cone-ordering, in general and therewith widely applicable theories. The range for applications of a theory depends of course strongly on the capacity of the notions in such a theory for fruitfull generalizations. Now I tend to say that a scientists' motives are to a large extent opportunistic and that a theory cannot be advocated only on whatever idealistic arguments, but instead of that its success depends mainly on answers to questions like: "leads it to a description of a present problem which is simplifying enough for handling?"; "suites it in scientific tradition so much that it admits meaningfull and operational generalizations of methods of the more restrictive theories of the past?" I would not pose these questions if for convex analysis I should not tend to an affinnative answer. In the next section a more concrete problem is discussed and attacked with easily accessible geometrical notions, of which we met some generalizations just now.

2. ON SOME CLASSICAL CONVEXITY CONCEPTS AND METHODS.

E^n denotes euclidean n-space, U a subset of E^n, C(U) its convex hull, i.e. the intersection of all convex sets of E^n, containing U. A polytope is the convex hull of a finite set. We shall not explicitly distinguish between E^n and its dual E^{n*} and exploit the innerproduct $(\ldots,\ldots)$ to describe linear functions and the metric as well. For didactical purposes we shall not hesitate to apply a non-essential transition to E^3 or E^2.

For $\varepsilon \geqq 0$ the so called ε-parallel-set U_ε of U is the union of all closed ε-balls around points of U,

(1) $$U_\varepsilon = U + \varepsilon B = (1-\varepsilon)U + \varepsilon U_1;$$

here B denotes the closed unit-ball (, Fig. 3). If U is compact convex, then also is U_ε. The volume V(U) of a compact convex set U is its

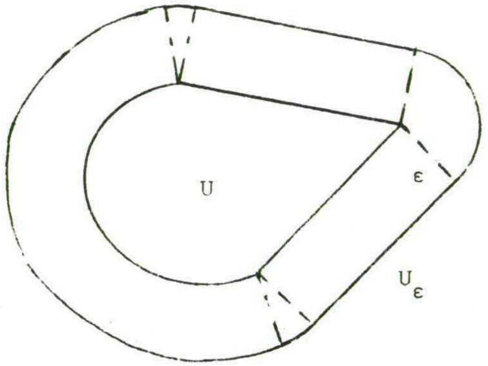

Fig. 3.

n-dimensional Lebesgue -measure and its surface-area is

$$F(U) := \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \cdot [V(U_\varepsilon) - V(U)]$$

The distance of two non-empty compact sets X, Y of E^n is

$$d(X,Y) := \inf \{\varepsilon | X \subset Y_\varepsilon \wedge Y \subset X_\varepsilon\};$$

d makes the family of compact convex subsets $\neq \emptyset$ of E^n a metric space (, Fig. 4.)

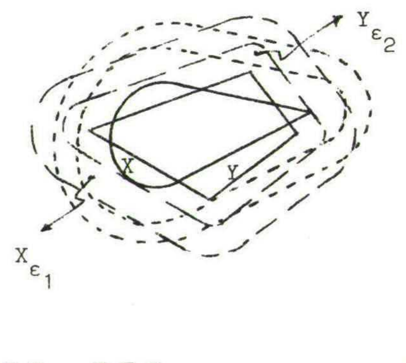

Fig. 4. $Y \subset X_{\varepsilon_1}$, $X \subset Y_{\varepsilon_2}$.

Of fundamental importance is the Blaschke selection-theorem: "from any infinite uniformly bounded family of compact convex sets a sequence can be selected which converges (with respect to the metric d) to a compact convex set. Particularly in approximation of convex sets by polytopes this theorem can often be used to guarantee the existence of a solution for extremal problems. Then a certain property is first proved for the polytopes and next by applying Blaschke's theorem for more general convex sets; of course the continuity of some set-functions, like measures, ought to be considered.

Another interesting procedure is Steiners' symmetrization. Be U compact with non-empty interior and H a hyperplane; the symmetrization SH(U) of U about H is the union of all segments on lines l orthogonal to H with their midpoints in H that are the translates of $l \cap H$ along l (, Fig. 5.). Main properties of SH are: i) SH(U) is compact convex

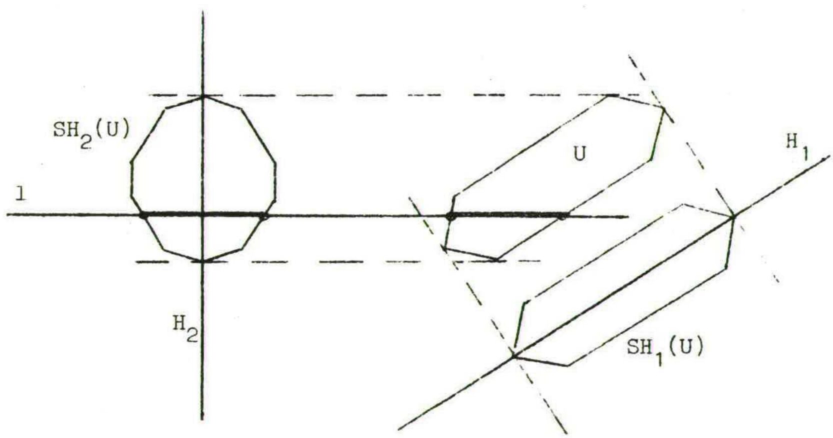

Fig. 5.

if U is; ii) V(SH(U)) = V(U); iii) F(SH(U)) $\leqq$ F(U) and the inequality-sign holds unless U is already symmetric about a hyperplane parallel H.

The first thing we want to point out is the "balling"-theorem: let U be compact convex with 0 as interior point, then from the family Σ of sets obtained from U by a finite sequence of successive symmetrizations about hyperplanes through 0 there can be selected a sequence that converges to a ball around 0. Let us sketch a proof. r(X) will denote the radius of the circumscribed sphere with centre 0 of X. Then there is a sequence $\{U_n\} \subset \Sigma$ with $r_0 = \lim_{n\to\infty} r(U_n)$ exists; by the selection theorem we can even take a sequence that converges itself to a compact convex set U_0. Let K be the ball with centre 0 and radius r_0 then obviously $U_0 \subset K$ and we shall show that $U_0 = K$. For if not so, then on the surface of K there is a closed spherical region T which has empty intersection with U_0 (, Fig. 6.). The entire surface of K

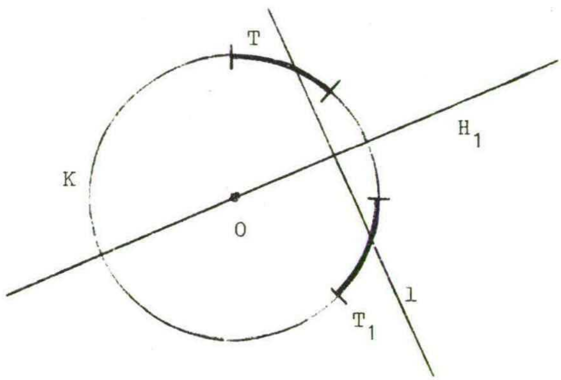

Fig. 6.

can be covered with a finite number $T_1, T_2, \dots, T_k$ of such regions congruent with T. If H_i is the hyperplane through 0 about which T and T_i are symmetric, then $SH_1(U_0)$ has no point in common with $T \cup T_1$ since for any $l \perp H_1$ the segment $l \cap U_0$ is shorter then the spherechord $l \cap K$. It follows that none of the sets $T, T_1, T_2, \dots, T_k$ does meet $\hat{S}(H) := SH_k(SH_{k-1}(\dots(SH_1(U_0))\dots)$, i.e. $\hat{S}(U_0)$ has void intersection with the surface of K. From the nature of U_0 as a limit of symmetrizations of U it is easily deduced that there exists a member of Σ with a positive distance to the surface of K and this is a contradiction with the fact that K has the infimum of radii of circumscribed spheres of members of Σ as radius.

We mentioned the fact that symmetrizations decreases generally the surface area. Since this is crucial we will give an idea of a proof for this fact and consider to avoid technicalities the 2-dimensional case. Let [a,b] be the projections of U on the x-axis (, Fig. 7.).

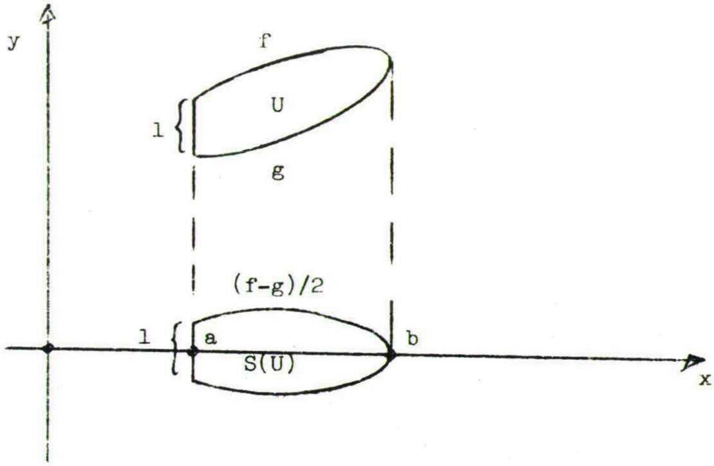

Fig. 7.

There are continuous functions f, g on [a,b] such that

$$U = \{(x,y) \mid a \leqq x \leqq b,\ g(x) \leqq y \leqq f(x)\}.$$

The symmetrization S(U) of U about the x-axis is simarly described with (f-g)/2 and -(f-g)/2 in stead of f,g. Now

$$F(U) = 1 + \int_a^b [\sqrt{1+f'^2} + \sqrt{1+g'^2}]\,dx,$$

$$F(S(U)) = 1 + \int_a^b 2\sqrt{1 + (\tfrac{f'-g'}{2})^2}\,dx.$$

The integrands are $\|\binom{1}{f'}\| + \|\binom{1}{-g'}\|$ and $\|\binom{2}{f'-g'}\|$, which are ordered by $\geqq$. The equality sign holds if and only if f' = -g'. So $F(U) \geqq F(S(U))$ and the inequality sign holds iff f' = -g' or f = -g + C almost everywhere on [a,b], and since f and g are continuous, everywhere; this means that U is symmetric about y = C/2.

Now we turn to things we partly already met in the first section.

$$U^{*} := \{f \in E^n \mid (f,u) \leqq 1,\ \forall u \in U\}$$

is called the polar of $U \subset E^n$. The polar of a set consisting of one point is a closed halfspace and U^* is therefore the intersection of the closed halfspaces that are the polars of its points (, Fig. 8.). Thus U^* is always closed and convex. Roughly speaking U^* accounts for the supporting-plane behaviour of U with respect to the location of 0. Some properties of U are dually reflected in U^*, e.g.

U is bounded $\Leftrightarrow$ 0 is an interior point of U^*.

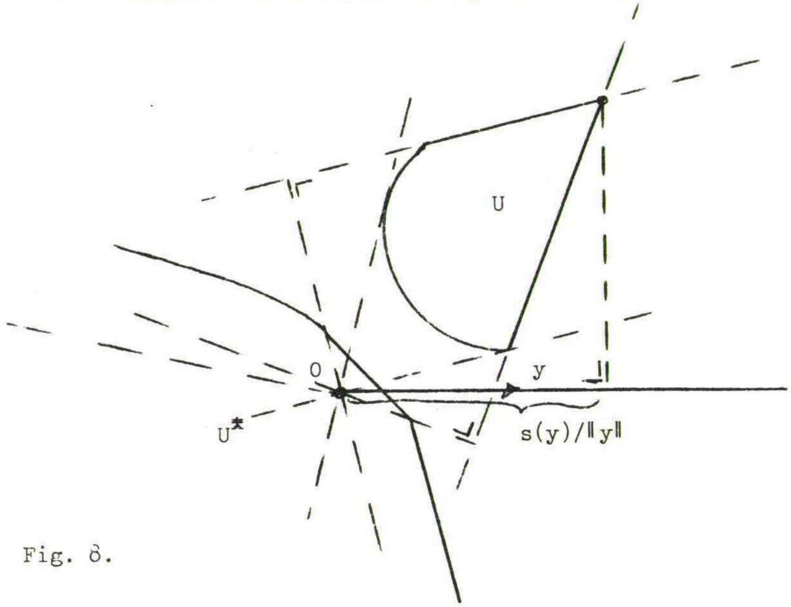

Fig. 8.

Then if U is compact convex consider the Minkowski-function of U^*

$$s(y) := \inf \{\lambda \mid \lambda > 0,\ y \in \lambda U^*\}$$

which is then defined on E^n and has the properties

(2) $\quad s(y) \geqq 0;\ s(\rho y) = \rho s(y);\ s(y+z) \leqq s(y) + s(z)$

for all $y, z \in E^n$ and all $\rho \geqq 0$, so that in particular s is a convex function. Because now

$$s(y) = \sup \{(y,u) \mid u \in U\}$$

s is mostly called the support-function of U (, Fig. 8). That a compact convex U can be recaptured from its support function,

$$U = \{u \mid (y,u) \leqq s(y),\ \forall y \in E^n\}$$

is a direct consequence of the separation property for convex sets. Denoting the support-function of U by s_U we have for $\alpha,\beta \geqq 0$ and bounded U,V

$$(3) \qquad s_{\alpha U+\beta V}(y) = \sup\ \{(y,u+v) \mid u \in V,\ v \in V\} = \alpha s_U(y) + \beta s_V(y).$$

We now show that certain boundary parts of a compact convex U have as support function the directional derivative of $s = s_U$ in an appropriate point. Let H be a supporting hyperplane of U, say $H : (a,x) = a_0$ where $(a,u) \leqq a_0$, $\forall u \in U$ and $a_0 = (a,r) = s(a)$ for some $r \in H \cap U$. Then $(y,r) \leqq s(y)$, $\forall y \in E^n$. The directional derivative of s in a in the direction v is

$$\delta s(a;v) := \lim_{\rho \downarrow 0} \frac{1}{\rho} [\, s(a+\rho v) - s(a)] .$$

Take $y = a + \rho v$, $\rho > 0$, then

$$(a,r) + \rho(v,r) = (a+\rho v,r) \leqq s(a+\rho v),$$

$$(v,r) \leqq \frac{1}{\rho} [\, s(a+\rho v)-s(a)] ,$$

$$(v,r) \leqq \delta s(a;v) \leqq \inf \frac{1}{\rho} [\, s(a+\rho v)-s(a)] \leqq$$

$$\frac{1}{\rho} [\, s(a)+\rho s(v)-s(a)] = s(v)$$

by a well-known monotonity property of the directional difference-quotient of a convex function and (2). Substitution of $v := a$ and $v := -a$ yields $(a,r) \leqq \delta s(a;a) \leqq s(a)$ and $-(a,r) \leqq \delta s(a;-a) = -s(a)$ so that $(a,r) = s(a) = \delta s(a;a)$. Now

$$\delta s(a;v) \geqq (v,r), \ \forall r \in H \cap U, \ \forall v \in E^n,$$

$$\delta s(a;a) = (a,r), \ \forall r \in H \cap U$$

means that $\delta s(a;v)$ as a function of the variable v is the support-function of the boundary-part $H \cap U$ of U.

If we now step over to E^3 and take for U a polytope with non-empty interior then the 2-dimensional faces of U have support-functions $\delta s(a_j;v)$ where the a_j are appropriate directed normal vectors to the hyperplanes containing those faces and the support-functions of the 1-dimensional faces of U are of the type $\delta^2 s(a_j;b_1;v)$ for appropriate vectors a_j, b_1 (, Fig. 9.).

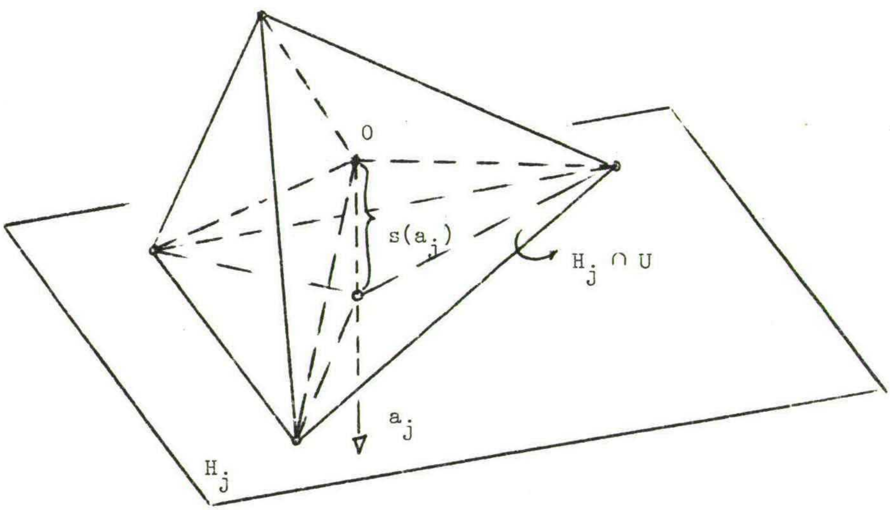

Fig. 9. $s(a_j)$ is the altitude from 0 to H_j

The 3-dimensional volume $V_3(U)$ of U is equal to

$$V_3(U) = \frac{1}{3} \sum_{j=1}^{k} s(a_j).V_2(H_j \cap U),$$

i.e. the algebraic sum of the volumes of the piramides on the faces of U with 0 as summit; here the a_j are unit normal vectors of the supporting hyperplanes H_j, pointing from U. Since the analogue holds for V_2 and so on, we get

$$V_3(U) = \frac{1}{3} \cdot \frac{1}{2} \cdot \sum_{j=1}^{k} \sum_{l=1}^{k_j} s(a_j)\delta s(a_j;b_l)V_1(H_{j,l} \cap U) =$$

$$= \frac{1}{3!} \sum_{j=1}^{k} \sum_{l=1}^{k_j} \sum_{p=1}^{k_{j,l}} s(a_j).\delta s(a_j;b_l).\delta^2 s(a_j,b_l;c_p).1,$$

where for consistency reasons the 0-dimensional volume of the vertices of U has been taken 1. Direct application of an approximation procedure towards general convex sets would lead to troubles well-known from the definition of multiple integrals. In stead we apply the formula to a linear combination $\alpha U + \beta V$ of polytopes, $\alpha,\beta \geqq 0$, which is again a polytope. Since the three support-functions in each term of the sum are homogeneous we see that $V_3(\alpha U + \beta V)$ is a third degree homogeneous polynomial in α,β, (cf. (3), (2)). Now by an approximation argument it can be proved that also for arbitrary compact convex sets $V_3(\alpha U + \beta V)$ is a third degree homogeneous polynomial in α,β. This then is a special case of the mixed volume representation by Minkowski. It is now obvious that

$$V(U_\varepsilon) = V(U + \varepsilon B) = A + B\varepsilon + C\varepsilon^2 + D\varepsilon^3;$$

and that $A = V(U)$ and $B = F(U)$. Already Steiner arrived at the formula

$$V(U_\varepsilon) = V(U) + F(U)\varepsilon + M\varepsilon^2 + \frac{1}{3} N\varepsilon^3,$$

where M and N are some curvature measures, with always $N = 4\pi$.
The famous Brunn-Minkowski theorem says that the function

$$f(\varepsilon) := [\,V(U_\varepsilon)]^{\frac{1}{3}} = [\,V((1-\varepsilon)U + \varepsilon U_1)]^{\frac{1}{3}}, \quad \varepsilon \geqq 0$$

is a concave-function; this result can be deduced by symmetrization and approximating somewhat along the lines we followed in proving the balling-theorem. Therefore $f''(\varepsilon) \leqq 0$, i.e.

$$-\frac{2}{9}[f(\varepsilon)]^{-5}.[(F^2-3MV)+(MF-3NV)\varepsilon+(M^2-NF)\varepsilon^2] \leqq 0;$$

so that

$$F^2 - 3MV \geqq 0,\ M^2 - NF \geqq 0$$

and by combination

$$F^3 \geqq 36\pi V^2.$$

This is the famous isoperimetric or isepiphanic inequality. Since equality holds for a ball, it implies that of all convex bodies with given volume V the ball has least area. By a similar argument of strict decreasing surface area on symmetrization as used before one can deduce that the ball is indeed the only solution of the isepiphanic problem among the convex bodies.

3. ON CONVEX SURFACES.

One might distinguish between a geometrical and analytical approach to convexity somewhat along the lines of an axiomatic- synthetic approach and an algebraic-analytical approach to geometry.
The double approach can deepen our insight considerably and reveal thus-far overseen aspects. With regard to convexity the geometric theory and the theory of convex functions are strongly connected. In the last section convex functions and analysis came in in a geometric extremal problem. On the other side the standard (convex) programming problem for instance confronts us via the constraints with questions about sublevel sets $\{x|\varphi_i(x) \leqq b_i\}$ and particularly with their boundaries. Fenchel, [6], occupied himself with the characterization of the convexity of such sets by means of quasi-convex functions.

The crucial place of extremal sets of (the boundary of) sets of feasible solutions or of solutions of systems of inequalities, [13], makes that we want to gain insight in convex surfaces. One might cherish a hope to get answers from that part of mathematics where manifolds for their intrinsic nature are investigated, i.e. differential geometry. A problem then is that the far developped theory of smooth manifolds is not of immediate use, since the main interest will often be directed towards non-regular parts of the surfaces. However, some smoothening procedure for a present surface suggests itself so that e.g. the regular curvature behavior of the smoothened surfaces gives by backward limiting the wanted information. A more direct synthetic approach to (convex) surface theory that also covers non-regularities has been advocated by Alexandrow, [1] , Busemann, [4] , a.o.

A surface S is in a very general way conceived as a metric space with metric d. Curves on S are continuous images in S of [0,1] and can be given a length that may be + ∞. S (, arcwise connected,) is called a surface with intrinsic metric iff for each pair $p,q \in S$ the infimum of the lengths of all curves on S joining p and q is equal to $d(p,q)$; this contrasts the approach to a metric via an infinitesinal line-element as in Riemannian geometry. An intrinsic metric happens to be present of course if shortest lines, i.e. curves realizing the distance on S, do exist. Shortest lines and geodetic lines on S are then immediate available tools for the characterization of convexity of S and on S.

For a smooth n-dimensional surface S there is associated to each point of it the tangent-space, being an n-dimensional linear space. In differential geometry one investigates S by studying the collection of the tangent spaces to S in points of S and their connexions. For not neccessarily smooth but convex surfaces the role of the linear tangent spaces ought to be replaced by that of tangent cones, so that it comes to ones mind to ask wether the study of a tangent cone bundle and a tangent dual cone bundle is an appropriate mean for convex surfaces.

I may end with an other speculative remark . The collection of tangent spaces of a surface S constitutes a multifunction on S, associating with each point of S its tangent space as a set. Given a so called connexion on S an inner differentiation-proces for S can be defined. Since (convex) multifunctions are applied in mathematical economics it seems for that purpose worthwhile to develop an appropriate differential calculus for them with an eye on differential geometric results.

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ECONOMIC THEORY AND DUALITY

by

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1. Introduction.

Convex analysis plays an important role in mathematical economics. Fundamental concepts as optimum and Nash-equilibrium are characterized by separation of convex sets. It will be shown, therefore, that these concepts are closely related (section 2, 3 and 5).
Separation of convex sets implies the existence of separating hyperplanes which again can be identified with "prices" (or covectors in the dual space). The theory of duality analyses the relation between sets, functions or correspondences, and their dual characterizations in terms of prices.
For several years, the authors have worked on duality[1)] in relation to equilibria and other economic concepts. Their earlier results are used in this paper, and new applications are introduced.
In Section 4, a summary is given on duality theory; the sections 6-9 contain some important applications. First, collective decisions and public goods are considered. Then we analyse the competitive equilibrium concept, and show how it may be reformulated as a Nash equilibrium in several ways. In sections 8 and 9, finally, dynamic economic systems are analyzed from this point of view, using a new criterion for optimality for an economy with consumption.

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1) A classical application of duality in economic theory is the so called indirect utility function. Later, Shephard (1970) has applied this concept in production theory. A survey is given by Diewert (1974).

2. Optimum for a single decision maker.

Let $X \subset \mathbb{R}^n$ be the choice set for a decision maker or agent. $\mathbb{R}^n$ will be called the action space of that agent (e.g., the consumption space if the agent is a consumer who considers all conceivable consumption bundles).
The preferences of the agent are given by a strong preference correspondence $P: X \to X$, such that $P(x)$ is the set of actions (elements) of X which are better than x. By assumption is $x \notin P(x)$, i.e. P is irreflexive[1].

Let $Y \subset \mathbb{R}^n$ be a constraint set of the agent. The triple $E := (X,P,Y)$ is said to be an (abstract) economy with a single agent.

Definition 2.1. Given the abstract economy $E := (X,P,Y)$, an action $x \in \mathbb{R}^n$ is called an optimum if it is:
(1) feasible: $x \in X \cap Y$,
(2) maximal : $P(x) \cap Y = \emptyset$.

A simple example of an abstract economy is given by the following Linear Programming problem:

$$\max ax,$$
$$\text{subject to: } b_k x \leqq c_k, \quad k = 1,2,\ldots,m$$
$$x \geqq 0 .$$

Then: $X := \mathbb{R}^n_+$

$P := \{y \in X \mid ay > ax\}$

$Y := \{x \in \mathbb{R}^n \mid b_k x \leqq c_k \;\; k = 1,2,\ldots,m\}$.

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1) If preferences are given by a weak correspondence R (which is also reflexive), a strong preference correspondence P can be der~~r~~ived from it: $P(x) := R(x) \setminus R^{-1}(x)$.

Proposition 2.2. Let $E := (X,P,Y)$ be such that: $[X \cap Y]$ is compact, convex and nonempty; P has an open graph $\{(x,y) | y \in P(x)\}$ in $X \times X$, and $x \notin \text{Conv } P(x)$[1].
Then there exists an optimum.
If also $x \in \text{Cl } P(x)$ for all $x \in X$ (local non-satiation), then $x \in \text{Bnd } Y$.

Proof: Define $\hat{P}(x) := \text{Conv } P(x)$; $\hat{P}$ also has an open graph and $x \notin \hat{P}(x)$.
Clearly, $\hat{P}(x) \cap Y = \emptyset$ implies $P(x) \cap Y = \emptyset$.
Suppose there exists no optimum with respect to $\hat{P}$, then $F(x) := \hat{P}(x) \cap Y \neq \emptyset$, for all $x \in X \cap Y$. Since F is lower hemi continuous in $X \cap Y$, there exists a continuous selection $f: [X \cap Y] \to [X \cap Y]$ (Michael, 1956).
According to Brouwer's fixed point theorem, there exists an action $x = f(x) \in F(x)$. This contradicts $x \notin \hat{P}(x)$.
Suppose further that $x \in \text{Cl } P(x)$ and $x \in \text{Int } Y$, then $P(x) \cap Y \neq \emptyset$, and x cannot be an optimum. □

Local non-satiation implies that the restrictions on the actions space given by the set of feasible actions are actually working or active. If some action x is an optimum, then it is on the boundary of both $\text{Cl } P(x)$ and Y, although $P(x) \cap Y = \emptyset$. Therefore, the convex sets $\text{Conv } P(x)$ and Y are separated by a hyperplane

$$H(p,\alpha) := \{y \in \mathbb{R}^n | py = \alpha\},$$

which contains x.
The theory of duality (as it will be developed in section 4), is based on this separation property.
Since a hyperplane is determined by a linear function and a scalar, it is possible to associate with each optimum (given local non-satiation and thus scarcity) at least one linear function p and a scalar α. If x is an action in an economy E, p can be interpreted as a price, and α

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1) Notations and definitions of a.o. operations on sets are given in section 4.

as a value of x at price p.

Definition 2.3. Let x be an optimum in an economy E, in which local nonsatiation prevails[1]. The linear function p which determines a separating hyperplane, given the scalar α, between $P(x)$ and B is said to be a characteristic price of the action x at value α.

At a characteristic price, any feasible action costs not more than α, and any better actions costs not less than α (and more than α if $x \in \text{Int } X$). In case of local nonsatiation, any optimum will cost exactly α.

The problem of defining an optimum in E will be put in a more general context, although no reasonable interpretation is given for a single decision maker.

Let $X \subset \mathbb{R}^n$ be the choice set on which a strong preference relation P is defined, as above. Let $Y \subset \mathbb{R}^n$ be the planning set in the action space $\mathbb{R}^n$, determining the set of actions which are eventually or conditionally feasible. The set of feasible actions are given by a constraint correspondence $B: [X \cap Y] \to [X \cap Y]$.
The quadruple (X,P,Y,B) is a single agent abstract economy and denoted by E_1.

Definition 2.4. An action x is said to be an optimum in E_1, if:
(1) $x \in X \cap B(x)$
(2) $P(x) \cap B(x) = \emptyset$.

1) If one assumes that $x \notin \text{Conv } P(x)$, as in proposition 2.2, the preference relation in E does not need to be convex.

3. Nash-equilibrium for a set of agents.

The economy will be extended to a set $I := \{1,2,\ldots,H\}$ of agents, each agent i having his own action space $\mathbb{R}^{n_i}$, in which his choice set X_i is defined.

The collective choice set of all agents is defined by:

$X^H := X_1 \times X_2 \times \ldots \times X_H \subset \mathbb{R}^N$, with $N := \sum_{i \in I} n_i$.

For each agent $i \in I$, a strong preference correspondence $P_i : X^H \to X_i$ associates to each $x^H := (x_1, x_2, \ldots, x_i, \ldots, x_H)$ a set of actions in X_i that are considered better to agent i than his action x_i in x^H, given also the actions of other agents in x^H.

Let $Y_i \subset \mathbb{R}^{n_i}$ be the planning set of agent $i \in I$, and $Y^H := Y_1 \times Y_2 \times \ldots \times Y_H \subset \mathbb{R}^N$ be the collective planning set of all agents.

The constraint correspondence $B_i : [X^H \cap Y^H] \to [X_i \cap Y_i]$ associates to each y^H a set of actions that are both eligible and feasible for agent i, given the planned actions of all agents. In a static model it seems reasonable to let depend $B_i(y^H)$ only on y_j, for $j \neq i$; formally, however, $B_i(y^H)$ may also depend on y_i.

The collective constraint correspondence $B^H : [X^H \cap Y^H] \to [X^H \cap Y^H]$ is defined by $B^H := \prod_{i \in I} B_i$.

An abstract economy with H agents is defined by:

$$E_H := (X_i, P_i, Y_i, B_i).$$

Definition 3.1. An action $x^H \in \mathbb{R}^N$ is called a Nash-equilibrium in E_H, if:

(1) $x^H \in \quad B^H(x^H)$;

(2) $\forall i \in I$: $P_i(x^H) \cap B_i(x^H) = \emptyset$.

If there is only one agent in E_H, then a Nash-equilibirum clearly coincides with an optimum in E_1, as defined in 2.4.

Theorem 3.2. (Shafer and Sonnenschein, 1976).

Let $E_H := (X_i, P_i, Y_i, B_i)$ be such that:

(a) $X^H \cap Y^H$ is a compact, convex and nonempty set;

(b) B_i is a continuous correspondence, $\forall i$;

(c) $B_i(x^H)$ is compact, convex and nonempty, $\forall x^H \in X^H \cap Y^H$;

(d) P_i has an open graph in $X^H \times X_i$;

(e) $x_i \notin \text{Conv } P_i(x^H)$, $\forall x^H \in X^H$.

Then there exists a Nash equilibrium.

If also for all i, $x_i \in \text{Cl } P_i(x^H)$, $\forall x^H \in X^H$, then $x_i \in \text{Bnd } B_i(x^H)$.

Proof[1]: Define $\hat{P}_i(x^H) := \text{Conv } P_i(x^H)$, and a correspondence $F_i: [X^H \cap Y^H] \to [X_i \cap Y_i]$ by

$$F_i(x^H) := \hat{P}_i(x^H) \cap B_i(x^H).$$

Let $Z_i := \{x^H \in X^H \cap Y^H | F_i(x^H) \neq \emptyset\}$.

Since $\hat{P}_i$ and B_i are l.h.c., Z_i is an open set.

The correspondence $F_i: Z_i \to X_i \cap Y_i$ is l.h.c. and therefore contains a continuous selection $f_i: Z_i \to X_i \cap Y_i$ with $f_i(x^H) \in F_i(x^H)$.

Define $G_i: X^H \cap Y^H \to X_i \cap Y_i$ by:

$$G_i(x^H) := \begin{cases} f_i(x^H) & \text{if } x^H \in Z_i \\ B_i(x^H) & \text{if } x^H \notin Z_i. \end{cases}$$

$G := \prod_I G_i$, is an uhc correspondence from $X^H \cap Y^H$ into itself and satisfies Kakutani's conditions for the existence of a fixed point $x^H \in G(x^H)$.

---

1) Shafer and Sonnenschein give another proof. The idea of this proof has been borrowed from Gale and Mas Collel (1975). A generalization of the theorem is given by J. Greenberg (1977).

This point is a Nash equilibrium, because by definition is $x_i \in B_i(x^H)$ and $x^H \notin Z_i$.
For, if $x^H \in Z_i$, then $x_i \in F_i(x^H) = \hat{P}_i(x^H) \cap B_i(x^H)$, which contradicts the irreflexivity of $\hat{P}_i$.
Finally, suppose $x \in \mathrm{Cl}\ P_i(x^H)$ and $x \in \mathrm{Int}\ B_i(x^H)$, then $\hat{P}_i(x^H) \cap B_i(x^H) \neq \emptyset$, contradicting condition (2) of definition (3.1).□

Again it is possible to associate with a Nash equilibrium x^H (assuming local nonsatiation) H price vectors $p_i \in R^{n_i^*}$ such that $H(p_i;\alpha)$ separates $B_i(x^H)$ and Conv $P_i(x^H)$, with the same interpretation as in section 2 above.

<u>Definition 3.3.</u> Let x^H be a Nash-equilibrium in E_H, in which each agent is locally not satiated. A set of <u>Nash-equilibrium prices</u> is composed of a characteristic price p_i of the action x_i at value α_i, for each agent $i \in I$.

4. Duality operations on sets and correspondences.

The duality concepts which are used here, are based on the notion of separation of sets in $\mathbb{R}^n$. A typical separation theorem gives necessary and sufficient conditions for the existence of a hyperplane separating two sets. Such a hyperplane divides $\mathbb{R}^n$ into two half spaces, each of which contains one set mentioned above.
Let X be a set in $\mathbb{R}^n$ and let a hyperplane be called a <u>bounding hyperplane</u> for X, if one halfspace associated with that hyperplane contains X. The set X can be characterized (and perfectly if X is convex) by the set of all bounding hyperplanes to X.
Since each hyperplane in $\mathbb{R}^n$ can be represented by a vector $p \in \mathbb{R}^{n*}$, and a scalar $\alpha \in \mathbb{R}$:

$$H(p;\alpha) := \{x \in \mathbb{R}^n | p\ x\ = \alpha\},$$

it is fruitful to consider the set of all linear functions having their domain in $\mathbb{R}^n$ and values in $\mathbb{R}$. This set is again a real euclidian n-space, denoted by $\mathbb{R}^{n*}$ and called the <u>dual space</u> of $\mathbb{R}^n$. The spaces $\mathbb{R}^n$ and $\mathbb{R}^{n*}$ are isomorphic and do not need to be distinguished, but a distinction is quite sensible if this theory is applied on economics. The primal space $\mathbb{R}^n$ will be identified with the <u>quantity</u> (or action) space, the dual space $\mathbb{R}^{n*}$ with the <u>price</u> or <u>valuation</u> space.

Let $H(p;\alpha) := \{x \in \mathbb{R}^n | px = \alpha\}$

$$H_-(p;\alpha) := \{x \in \mathbb{R}^n | px \leqq \alpha\}$$

$$H_+(p;\alpha) := \{x \in \mathbb{R}^n | px \geqq \alpha\},$$

and $H(p) := H(p;1)$, $H_-(p) := H_-(p;1)$, $H_+(p) := H_+(p;1)$.
The idea expressed above that a set $X \subset \mathbb{R}^n$ can be characterized by the set of all bounding hyperplanes to X, is made concrete by means of the dual relation between hyperplanes in $\mathbb{R}^n$ and prices (covectors) in $\mathbb{R}^{n*}$:[1)]

<u>4.1. Polar cones, sets, and correspondences.</u>

<u>Definition 4.1.</u> Let X be a set in $\mathbb{R}^n$. The <u>polar sets</u> of X are defined by:

$$X_-^* := \{p \in \mathbb{R}^{n*} | X \subset H_-(p)\} = \{p \in \mathbb{R}^{n*} | \forall x \in X: px \leq 1\}$$

$$X_+^* := \{p \in \mathbb{R}^{n*} | X \subset H_+(p)\} = \{p \in \mathbb{R}^{n*} | \forall x \in X: px \geq 1\}$$

and are called the negative-polar, resp. positive-polar set.

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1) Since this distinction is mathematically not necessary in case of a finite eucledean space, all properties derived here can be applied both in $\mathbb{R}^n$ and $\mathbb{R}^{n*}$. Also $(\mathbb{R}^{n*})^* = \mathbb{R}^n$.

The polar cones of X are defined by:

$$X_-^0 := \{p \in \mathbb{R}^{n*} \mid X \subset H_-(p;0)\} = \{p \in R^{n*} \mid \forall x \in X\colon\ px \leq 0\}$$

$$X_+^0 := \{p \in \mathbb{R}^{n*} \mid X \subset H_+(p;0)\} = \{p \in R^{n*} \mid \forall x \in X\colon\ px \geq 0\}$$

and are called the negative- resp. positive-polar cone of X.

The positive-polar set X_+^* contains all $p \in \mathbb{R}^{n*}$ such that the hyperplanes H(p) separate X and {0}, see fig. 4.a.
The negative-polar set X_-^* contains all $p \in \mathbb{R}^{n*}$ such that X and {0} are on one side of H(p). The polar cones contain all p such that the hyperplanes H(p) have X on the negative, resp. positive side.

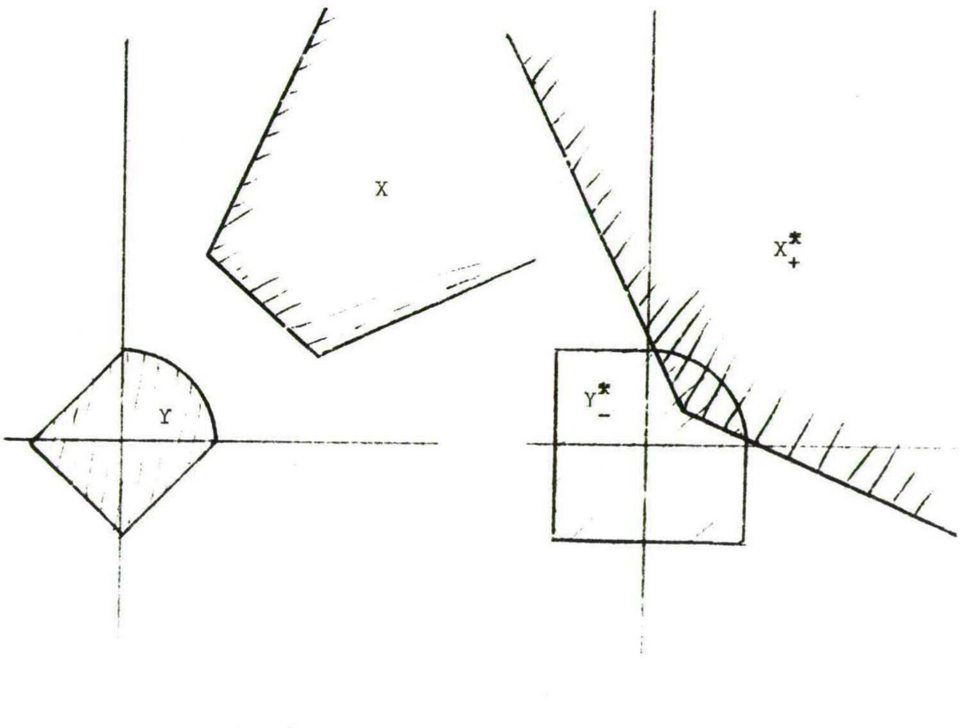

fig. 4.a. Polar sets.

The following hull (or closure) operations are defined:

Definition 4.2. Let $X \subset \mathbb{R}^n$.

$\text{Aff } X := \{x \in \mathbb{R}^n \mid x = \sum_{i=1}^{n+1} \lambda_i x_i,$ for $x_i \in X$ and $\Sigma\lambda_i = 1\}$:
the affine hull of X.

$\text{Conv } X := \{x \in \mathbb{R}^n \mid x = \sum_{i=1}^{n+1} \lambda_i x_i,$ for $x_i \in X$ and $\Sigma\lambda_i = 1,\ \lambda_i \geqq 0\}$:
the convex hull of X.

$\text{Aur } X := \{x \in \mathbb{R}^n \mid x = \lambda y,$ for $y \in X$ and $\lambda \geqq 1\}$:
the aureole hull of X.

$\text{Star } X := \{x \in \mathbb{R}^n \mid x = \lambda y,$ for $y \in X$ and $0 \leqq \lambda \leqq 1\}$:
the star hull of X.

$\text{Cone } X := \{x \in \mathbb{R}^n \mid x = \lambda y,$ for $y \in X$ and $\lambda \geqq 0\}$:
the cone closure of X.

$\text{Norm}_K\, X := (X-K) \cap K$, for cone K:[1)]
the normal hull with respect to K

$\text{Mon}_K\, X := (X+K) \cap K$, for cone K:[1)]
the monotone hull wrt K.
with respect to

Sets which are equal to their hull are called to be accordingly: affine, convex, etc.

The following "opening" operation is used:

Definition 4.3. Let $X \subset \mathbb{R}^n$.

$\text{Conint } X := \{x \in \mathbb{R}^n \mid \forall\lambda \geqq 0:\ \lambda x \in X\}$:
the interior cone of X.

---

1) If $K = \mathbb{R}^n_+$, no mention is made of K in the notation.

Definition 4.4.

Apart from the usual operations on sets (λX, $X+Y$, $X \cap Y$, $X \cup Y$), we shall use an operation called dual addition, convex intersection, or inverse addition (Rockafellar 1970):

$$\bigcup_{\lambda \in [0,1]} [\lambda X \cap (1-\lambda)Y] .$$

In general, if $X_i \subset \mathbb{R}^n$ are sets, $i \in I := \{1,2,\ldots,n\}$ and $L := \{\lambda \in \mathbb{R}^n \mid \Sigma\lambda_i = 1 \text{ and } \lambda_i \geqq 0\}$, then the expression becomes:

$$\bigcup_L \bigcap_I \lambda_i X_i .$$

Sometimes we use the notation[1]:

$$X \mathring{\cap} Y := \bigcup_{\lambda \in [0,1]} [\lambda X \cap (1-\lambda)Y] .$$

The following properties of polar sets are given without proof; their proofs, or references to their proofs, can be found in Weddepohl (1970, 1972, 1973), and Ruys (1972, 1974).

Property 4.5. (valid for both positive and negative polar operations: suffix is therefore omitted).

1. $X^* = (\text{Rint } X)^* = (\text{Cl } X)^* = (\text{Conv } X)^*$.
2. X^* is closed and convex.
3. $p \in \text{Bnd } X^* \Leftrightarrow H(p,1)$ supports X.
4. $X \subset Y \Leftrightarrow X^* \supset Y^*$.
5. $(\lambda X)^* = \lambda^{-1} X^*$, for $\lambda > 0$.
6. $(X \cup Y)^* = (X \cap Y)^*$.

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1) Analogously, one may define an operation convex addition:

$$X \mathring{+} Y := \bigcup_{\lambda \in [0,1]} [\lambda X + (1-\lambda)Y] = \text{Conv } [X \cup Y] .$$

7. $(X \cap Y)^{\star} = \mathrm{Conv}(X^{\star} \cup Y^{\star})$

8. $(X + Y)^{\star} = \mathrm{Cl}\ [\bigcup_{\lambda \in [0,1]} (\lambda X^{\star} \cap (1-\lambda)Y^{\star})]$.

Property 4.6. (on negative polar sets).

1. $0 \in \mathrm{Int\ Conv}\ X \Leftrightarrow X_{-}^{\star}$ is bounded.
2. $X_{-}^{\star} = (\mathrm{Star}\ X)_{-}^{\star}$.
3. $X_{-}^{\star} = \mathrm{Star}(X_{-}^{\star})$, and thus contains 0.
4. $X_{-}^{\star} \supset X_{-}^{0} = \mathrm{Conint}(X_{-}^{\star}) = (\mathrm{Cone}\ X)_{-}^{\star} = (\mathrm{Cone}\ X)_{-}^{0}$.
5. $X \subset K \Rightarrow [X_{-}^{\star} \cap K_{+}^{0}]$ is K_{+}^{0}-normal, for some K. $(X_{-}^{\star} - K_{+}^{0}) \cap K_{+}^{0} = X_{-}^{\star}$.

Property 4.7. (on positive polar sets).

1. $0 \in \mathrm{Cl\ Conv}\ X \Leftrightarrow X_{+}^{\star} = \emptyset$.
2. $X_{+}^{\star} = (\mathrm{Aur}\ X)_{+}^{\star}$.
3. $X_{+}^{\star} = \mathrm{Aur}(X_{+}^{\star})$, and does not contain 0.
4. $X_{+}^{\star} \subset X_{+}^{0} = \mathrm{Cl\ Cone}(X_{+}^{\star}) = (\mathrm{Cone}\ X)_{+}^{\star} = (\mathrm{Cone}\ X)_{+}^{0}$.
5. $X \subset K \Rightarrow X_{+}^{\star} + K_{+}^{0} = X_{+}^{\star}$, i.e. K_{+}^{0}-monotone.

Property 4.8. (reflexivity conditions).
Let X be a closed and convex set. Then:

1. $[(X_{+}^{\star})_{+}^{\star} = X] \Leftrightarrow X$ is aureoled and $0 \notin X$.
2. $[(X_{-}^{\star})_{-}^{\star} = X] \Leftrightarrow X$ is starred (i.e. $0 \in X$).
3. $[(X_{+}^{0})_{+}^{0} = X] \Leftrightarrow X$ is a cone.
4. $[(X_{-}^{0})_{-}^{0} = X] \Leftrightarrow X$ is a cone.

Property 4.9. (dual separation theorem).
Let X be closed, convex, aureoled and not containing 0 (i.e. aureole-reflexive), and Y be closed, convex and containing 0 (i.e. star-reflexive), then:

4.2. Dual correspondences of superlinear and convex maps.

The idea of characterizing a (closed and convex) set by a set of linear functions or prices, and scalars or values can also be applied on functions and correspondences, in order to obtain characteristic functions or correspondences. The link between these maps and the sets of the previous section is the epigraph of a function, or the graph of a correspondence (or multifunction).
Although there is a one to one correspondence between (multi-)functions and (epi-)graphs, given the domain and range of the (multi-)functions, there are several possibilities to partition the space in which the polar (epi-)graph is defined into a domain and a range. Two possibilities are compared here, generating the so called conjugate, resp. adjoint operation.

Let f be a function from $\mathbb{R}^n$ into $\mathbb{R}$ such that its epigraph $\{(x,\mu) \in \mathbb{R}^{n+1} | \mu \geqq f(x)\}$ is convex. The <u>conjugate</u> of f, denoted by f^∇, is the pointwise supremum of all affine functions $h(x) := bx - \beta$; so $f^\nabla : \mathbb{R}^n \to \mathbb{R}$, defined by:

$$f^\nabla(b) := \sup_x \{bx - f(x)\}.$$

A typical bounding hyperplane to the epigraph of f is thus:

$$\{(x,\mu) \in \mathbb{R}^{n+1} | \exists b,\beta : bx - \mu = \beta \leqq f^\nabla(b)\} \text{ 1)},$$

which is denoted by:

$$H[(b,-1),\beta], \text{ for } \beta \leqq f^\nabla(b),$$

the covector in $\mathbb{R}^{(n+1)\star}$ being $(b,-1)$, and the scalar being β.

---

1) Notice that epi $f = \{(x,\mu) | \forall b: bx - \mu \leqq f^\nabla(b)\}$.

1. $[X \cap Y = \emptyset] \Rightarrow [X_+^* \cap Y_-^* \neq \emptyset.]$

2. If $[\mathrm{Cl\ Cone}\ X \cap \mathrm{Conint}\ Y] \subset \{0\}$, then:
$[(X \cap Y) \neq \emptyset$ and $(\mathrm{Rint}\ X \cap \mathrm{Rint}\ Y) = \emptyset] \Leftrightarrow$
$[(X_+^* \cap Y_-^*) \neq \emptyset$ and $(\mathrm{Rint}\ X_+^* \cap \mathrm{Rint}\ Y_-^*) = \emptyset.]$

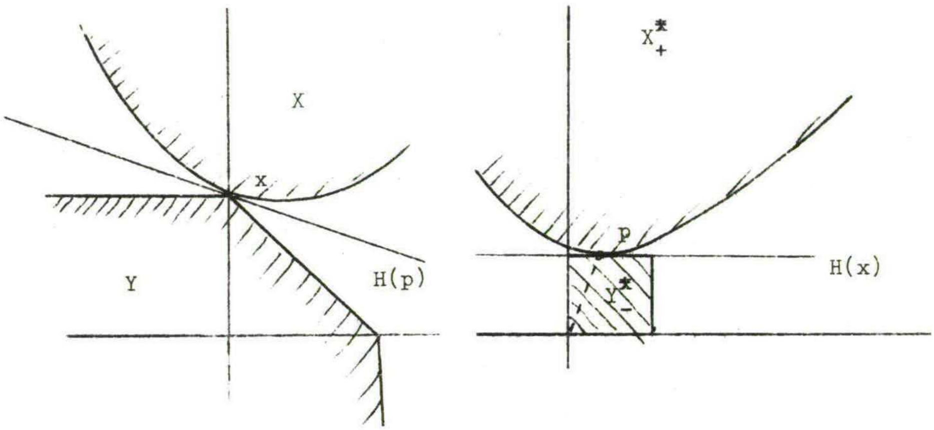

fig. 4.b.: dual separation

Property 4.10. (on polar correspondences).
Let $F: X \to Y$ be a correspondence.
If F is point-aureole-reflexive (i.e. point-closed, point-convex, point-aureoled and $0 \notin F(x)$, $\forall x$), then:

1. F is lhc $\Rightarrow F_+^*$ is closed.

2. [F is closed and for all $x \in X$ and for some neighborhood N of 0, $F_+^*(x) \cap N = \emptyset$,] $\Rightarrow F_+^*$ is lhc.

If F is point-star-reflexive (i.e. point-closed, point-convex and point-starred), then

1. F is lhc $\Rightarrow F_-^*$ is closed.

2. F is closed $\Rightarrow F_-^*$ is uhc.

The difference with the polarity operation $\star$ defined above is that here one component of the vector is fixed (on -1), instead of the scalar. Therefore, the principle is the same, only interpretations have to be adapted, and a choice can be made according to the circumstances.

The conjugate operation can also be applied on correspondences or multi-functions, as is done by Makarov and Rubinov (199, p. 145). In this paper, however, it will be fruitful to use the adjoint operation as defined by Ruys (1974, p. 191).

Let $F: \mathbb{R}^m \to \mathbb{R}^n$ be a correspondence. The correspondence F'_- and F'_+ from $\mathbb{R}^{n\star}$ into $\mathbb{R}^{m\star}$, defined by:

$$F'_-(q) := \{p \mid (-p,q) \in [\,Gr(F)]^{\star}_{-}\}, \text{ and}$$

$$F'_+(q) := \{p \mid (p,-q) \in [\,Gr(F)]^{\star}_{-}\},$$

are said to be the <u>lower and upper adjoint</u> of F, respectively.
If F is a linear function, then the adjoints coincide and correspond with the usual definition. If Gr(F) is a cone, then they correspond with Rockafellar's (1972) definition, e.g.:

$$F'_+(q) := \{p \mid \forall x, \forall y \in F(x): px \leqq qy\}.$$

But if Gr(F) is a convex set in $\mathbb{R}^{n+m}$, then they are equal to:

$$F'_-(q) := \{p \mid \forall x, \forall y \in F(x): qy \leqq px + 1\}, \text{ and}$$

$$F'_+(q) := \{p \mid \forall x, \forall y \in F(x): px \leqq qy + 1\}.$$

However, since we will use a composition operation on correspondences, the scalar 1 cannot be fixed or choosen independently any more. Further, our attention will be restricted to superlinear and convex correspondences, defined in closed and convex cones. And finally, it may be easier to use the inverse of an adjoint, which has more resemblance with the primal correspondence. This leads us to the following definition.

Definition 4.11. Let $K_1 \subset \mathbb{R}^m$ and $K_2 \subset \mathbb{R}^n$ be closed, convex and solid cones, and $F\colon K_1 \to K_2$ be a correspondence.
The dual correspondence of F with respect to the positive scalar Π, $F^{\circledast}\colon (K_1^0 \times \mathbb{R}) \to K_2^0$, with $K_1^0 := (K_1)_+^0$ and $K_2^0 := (K_2)_+^0$, is defined by:

$$F^{\circledast}(p,\Pi) := \begin{cases} \{q \in K_2^0 \mid \forall x, \forall y \in F(x)\colon qy \leqq px + \Pi\}, & \text{if } 0 \in F(\text{Int } K_1), \\ \{q \in K_2^0 \mid \forall x, \forall y \in F(x)\colon px \leqq qy + \Pi\}, & \text{if } 0 \notin F(\text{Int } K_1). \end{cases}$$

Definition 4.12. Let K_1 and K_2 be closed, convex cones in $\mathbb{R}^n$, resp. $\mathbb{R}^m$, and F be a correspondence from K_1 into K_2.
F is said to be superlinear if it is:

1) superadditive: $F(x+y) \supset F(x) + F(y)$;

2) positive homogeneous: $F(\lambda x) = \lambda F(x)$, $\forall \lambda > 0$;

3) closed: $Gr(F)$ is closed in $K_1 \times K_2$;

4) a Gale map: $F(0) = \{0\}$;

5) nondegenerate: $F(K_1) \cap [\text{Int } K_2] \neq \emptyset$.

F is said to be a convex-star map, if its graph is a closed and convex set, if the cone closure and the cone interior of graph F meet the conditions on the graph of a superlinear correspondence[1)], and if $0 \in F(x)$ for all $x \in K_1$.

F is said to be a convex-aureole map, if its inverse is a convex-star correspondence.

---

1) These are: $Gr(F)$ is a closed, convex cone contained in $K_1 \times K_2$, such that $(0,y) \in Gr(F) \Rightarrow y = 0$, and $Pr_2\, Gr(F) \cap [\text{Int } K_2] \neq \emptyset$.

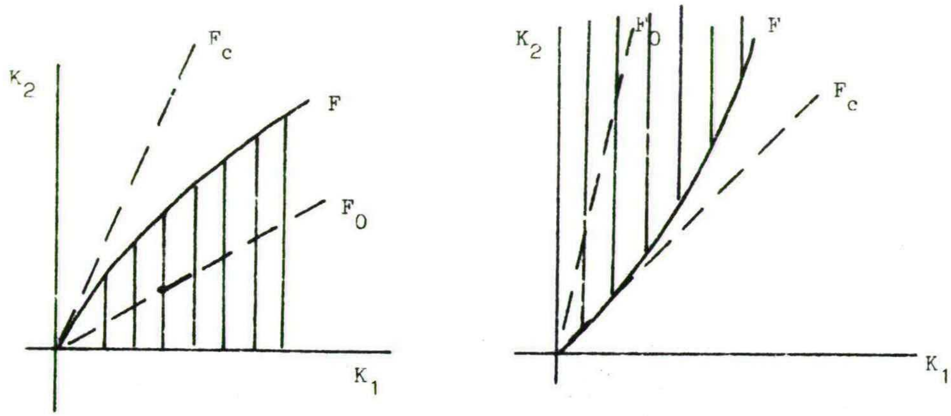

fig. 4.c.

It should be noticed that:

$$Gr(F^{\circledast}) = -[Gr(-F)]^{*}_{-}, \text{ if } F \text{ is a convex-star map;}$$

$$Gr(F^{\circledast}) = [Gr(-F)]^{*}_{-}, \text{ if } F \text{ is a convex-aureole map.}$$

(See also fig. 4.c)

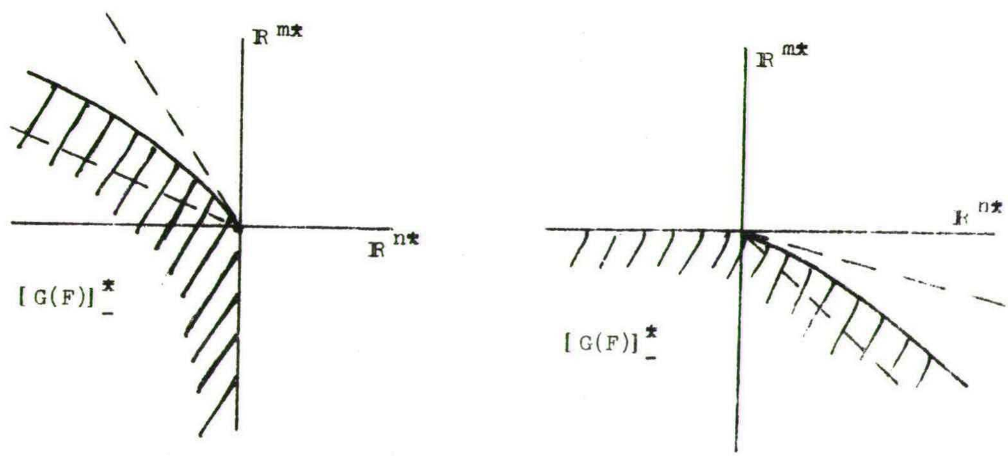

fig. 4.d.

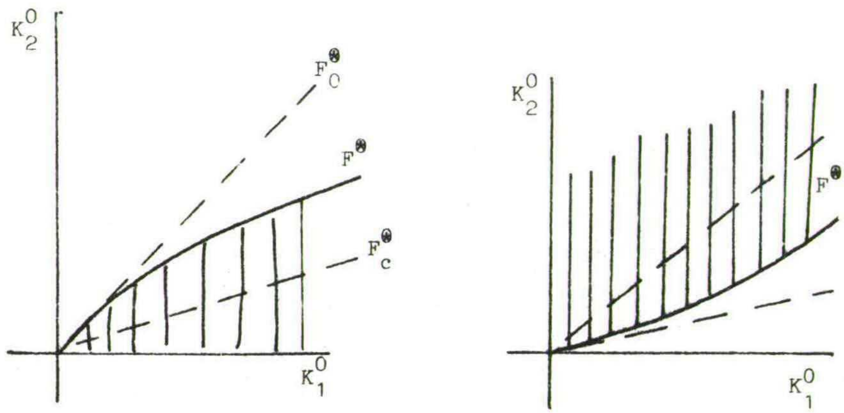

fig. 4.d.: polar graphs and dual correspondences of a convex-star, and convex-aureole map.

The correspondence $F_c\colon K_1 \to K_2$ defined by:

$$F_c(x) := \{y \mid (x,y) \in \text{Cone } Gr(F)\}$$

is called the <u>cone-closure</u> of F, and analogously the <u>cone-opening</u> of F, F_0, is defined. Then it is clear that F_c and $F_0^{\circledast}$ describe the behavior of F, resp. $F^{\circledast}$ near the origin, and F_0, resp. $F_c^{\circledast}$, the behavior of F, resp. $F^{\circledast}$ in the infinite. If F is a superlinear map, then evidently F, F_0 and F_c coincide.

The following properties of convex-star and convex-aureole correspondences can be derived (see Ruys, 1974):

Property 4.13. Let $F: K_1 \to K_2$ be a convex-star correspondence. Then:

1. F is point-compact, point-starred, and increasing: i.e.:

$$[x,y, \text{ and } (y-x) \in K_1] \Rightarrow [F(x) \subset F(y)].$$

2. $F^{\circledast}$ is a convex-star map, and also K_2^0- normal, i.e.:

$$\text{for all } p \in K_1^0\text{: } F^{\circledast}(p) = [F^{\circledast}(p) - K_2^0] \cap K_2^0.$$

3. (reflexivity) $[F^{\circledast\circledast} = F] \Leftrightarrow [F \text{ is } K_2\text{-normal}]$.
4. (composition) GoF is a convex-star map; $(G: K_2 \to K_3)$

$$(GoF)^{\circledast} = G^{\circledast} o F^{\circledast}.$$

5. F is uhc and lhc.
6. If Gr(F) is a cone, then F is superlinear.

Property 4.14. Let $F: K_1 \to K_2$ be a convex-aureole correspondence. Then:

1. F is point-aureoled and decreasing, i.e.:

$$[x,y \text{ and } (y-x) \in K_1] \Rightarrow [F(x) \supset F(y)].$$

2. $F^{\circledast}$ is a convex-aureole map, and also K_2^0-monotone, i.e.:

$$\text{for all } p \in K_1^0\text{: } F^{\circledast}(p) = [F^{\circledast}(p) + K_2^0] \cap K_2^0.$$

3. (reflexivity) $[F^{\circledast\circledast} = F] \Leftrightarrow [F \text{ is } K_2\text{-supernormal}]$.
4. (composition) GoF a convex-aureole correspondence, if both F and $G: K_2 \to K_3$ are so;

$$(GoF)^{\circledast} = G^{\circledast} o F^{\circledast}.$$

5. F is lhc.

Property 4.15. (duality).

1. If F is a convex-star map, then:

$$\max_{y \in F(x)} qy = \inf_{p \in (F^{\circledast})^{-1}(q)} px + 1.$$

2. If F is a convex-aureole map, then:

$$\inf_{y \in F(x)} qy = \max_{p \in (F^{\circledast})^{-1}(q)} px - 1.$$

4.3. Polar and dual preferences, and optimum.

Polarity operations can be applied on the concepts defining an optimum in an economy E (definition 2.1):

Assume that X in $\mathbb{R}^n$ is closed, convex and aureoled[1] and does not contain 0; P(x) is convex and aureoled[1], $x \notin P(x)$ and $x \in \text{Cl } P(x)$, for all $x \in X$; P has an open graph, and Y is closed, convex and starred.

Then $[P(x)]^{\star}_{+}$ is closed in $\mathbb{R}^{n\star}$, convex and aureoled.
Following the notation introduced in 4.1, this correspondence should be indicated by $P^{\star}$. However, in order to avoid confusion with the strong character given to a preference relation P (in contrast to R), we will denote $[P(x)]^{\star}$ by $\bar{P}^{\star}$. The correspondence $\bar{P}^{\star}: X \to \mathbb{R}^{n\star}$ has a closed graph.

If x is an optimum, then it follows from property 4.9 that:

$$\bar{P}^{\star}(x) \cap Y^{\star} \neq \emptyset \text{ and Rint } \bar{P}^{\star}(x) \cap Y^{\star} = \emptyset. \text{ [2]}$$

---

1) If X and P(x) are not aureoled, we could replace them by Aur X and Aur P(x); Y may be replaced by Star Y. Under certain conditions, this will not affect the optimum.

2) Or, equivalently, Rint $\bar{P}^{\star}(x) \cap$ Rint $Y^{\star} = \emptyset$.

The first intersection contains the optimum prices (hence $H(p)$ separates $P(x)$ and Y), and $H(x)$ separates $\bar{P}^{\star}(x)$ and $Y^{\star}$.

Next, we will define a preference correspondence in terms of prices.

Let $X \subset \mathbb{R}^n$ be a closed, convex, aureoled set, not containing 0. Assume that the preferences are given by a weak preference correspondence, $R\colon X \to X$, for which the following assumptions hold: $\forall x,y \in X$:

1. $y \in R(x)$ or $x \in R(y)$: completeness.
2. $R(x) \subset R(y)$ or $R(y) \subset R(x)$: transitivity.
3. $R(x)$ is convex.
4. $R(x)$ is aureoled.
5. $R(x)$ and $R^{-1}(x)$ are closed.
6. $\lambda < 1 \Rightarrow \lambda x \notin R(x)$: monotonicity.

These assumptions imply that:

a. $P(x) = \text{Int } R(x) = R(x) \setminus R^{-1}(x)$.

b. R has a closed graph.

c. R is lhc.

d. P has an open graph.

Let for some (well chosen) $x_0 \in X$,

$$V := R(x_0).$$

The correspondence $\hat{R}^{\star}\colon V^{\star} \to V^{\star}$ defined by:

$$\hat{R}^{\star}(p) := \bigcap_{x \in R^{\star-1}(p)} R^{\star}(x),$$

is said to be a weak preference relation in terms of prices. In case of private goods, $q \in R^0(p)$ is to be interpreted as: "at price q, and given value 1, no bundle can be bought that is better than some bundle which can be bought at price p".

The relation $R^{\star}$ is a complete, transitive, point-convex, point-aureoled, point-closed, lhc, and such that $\lambda < 1$ implies $\lambda p \notin \hat{R}^{\star}(p)$. The inverse $(R^0)^{-1}$ does not need to be closed[1], hence $\hat{R}^{\star}(p)\setminus(\hat{R}^{\star})^{-1}(p)$ does not need to be open.
However, if we define $\hat{P}^{\star}: V^{\star} \to V^{\star}$ by

$$\hat{P}^{\star}(p) := \text{Int } \hat{R}^{\star}(p),$$

then $P^{\star}$ has an open graph.

4.4. Linear optimization and duality.

The purpose of this subsection is twofold: (a) it serves as an illustration of the theory of the preceding sections; (b) it shows the relation that exists between the concept of duality of this paper and the duality concept that occurs in optimization theory (dual linear program etc.).

In linear optimization models, preferences are given by a linear utility function $u(x) := ax$. The preference correspondence that follows from this function is $R(x) = \{\tilde{x} \in \mathbb{R}^n | a\tilde{x} \geqq ax\}$. Clearly this correspondence has all nice properties that we may need (continuous, point closed, convex, monotonous etc.) it also has a nice dual preference correspondence in terms of prices.
Two models are considered:
(1) linear constraints, but no sign constraint;
(2) linear constraints and the requirement that the solution should be non-negative (linear programming).

1) If R also meets the assumption:
$[H(p)$ supports (asymptotically) $R(x)$ and $R(y)] \Rightarrow [R(x) = R(y)]$,
then $(\hat{R}^{\star})^{-1}(p)$ is closed, and $\hat{P}^{\star}(p) := \text{Int } R^{\star}(p) = R^{\star}(p)\setminus(R^{\star})^{-1}(p)$.
In this case, $R^{\star}$ has also a closed graph.

(1) Consider the following problem:
max ax with constraints[1] $Ax \leq \ell$ for $x \in \mathbb{R}^n$, $a \in \mathbb{R}^{n*}$,
$\ell = \{1,1,\dots,1\} \in \mathbb{R}^m$ and A an n×m-matrix

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_m \end{bmatrix}, \text{ with } a_k \in \mathbb{R}^{n*}$$

The abstract economy $E = \{X,R,Y\}$ is now defined by[2]:

$$X := \{x \in \mathbb{R}^n | ax \geq 0\}$$

$$R(x) := \{\tilde{x} \in X | a\tilde{x} \geq ax\}$$

$$Y := \{y \in \mathbb{R}^n | Ay \leq \ell\}$$

R is a weak preference correspondence and

$$P(x) = \{\tilde{x} \in X | a\tilde{x} > ax\}.$$

In an optimum we have

$$P(x) \cap V = \emptyset \text{ and } x \in Y.$$

We define the dual economy $E^* = \{V^*, \hat{R}^*, Y^*\}$ by

---

1) Clearly any set of constraints $a_k x \leq b$, for which an $\bar{x}$ exists such that $a_k\bar{x} \leq b$ for all k, can be put in the required form by writing

$$a_k' = \frac{a_k}{b-a_k\bar{x}} \text{ and } x' = x-\bar{x}, \text{ hence } a_k' x' = \frac{a_k}{b-a_k\bar{x}} \leq \frac{b-a_k\bar{x}}{b-a_k\bar{x}} = 1.$$

2) X and V^* contain 0 in their boundary contrary to assumptions made in preceding sections. In this case this can do no harm.

$$V^{*} := \{p \in \mathbb{R}^{n*} | p = \lambda a,\ \lambda \geqq 0\}$$

$$R_{+}^{*}(p) := \{p \in V^{*} | \tilde{p} \geqq p\}$$

$$Y_{-}^{*} := \mathrm{Conv}\ \{a_1, a_2, \ldots, a_m, 0\}$$

Clearly

$$P^{*}(x) := \{\tilde{p} \in V^{*} | \tilde{p} > p\}$$

For an optimum price in E^{*}, we have

$$P^{*}(p) \cap Y^{*} = \emptyset \text{ and } p \in Y^{*}$$

Clearly p is colinear with a.

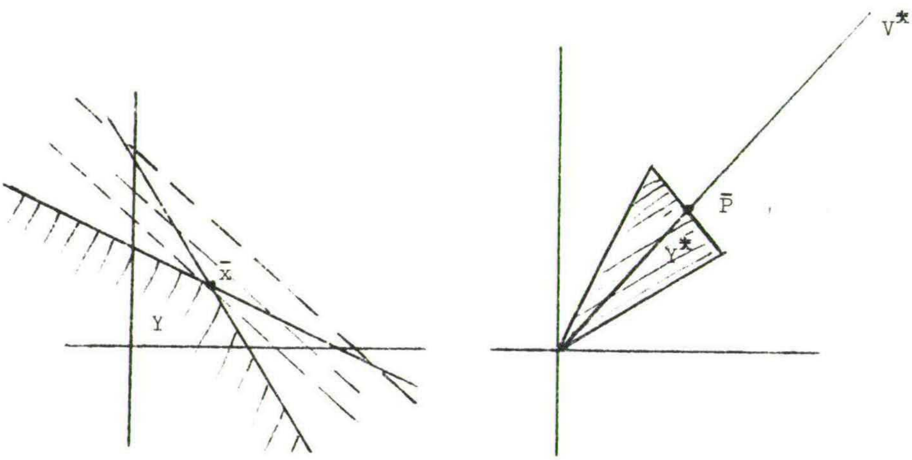

fig. 4.e.

This is equivalent to finding $p = \lambda a$, such that:

$$\lambda a \in Y^{*} \text{ and } \lambda' > \lambda \Rightarrow \lambda' a \notin Y^{*}.$$

Since we may write:

$$Y^{*} = \{p \in \mathbb{R}^{n*} \mid \exists r \in \mathbb{R}^{m*}: p = rA,\ r \geqq 0 \text{ and } r\ell \leq 1\},$$

we have that p is optimal if $p = \lambda a$, such that:

$$\lambda = \max\{\lambda' \in \mathbb{R}_{+} \mid \lambda a = rA,\ r \geqq 0,\ r\ell \leq 1\},$$

or equivalently: $p = \lambda a$ and

$$\frac{1}{\lambda} = \min\{\mu \in \mathbb{R}_{+} \mid a = qA,\ q \geqq 0 \text{ and } q\ell \leq \mu\}.$$

This precisely corresponds to the dual program of the original problem[1]:

$$\min q\ell, \text{ with constraints } qA = a \text{ and } q \geqq 0.$$

(2) The following problem is a true linear programming problem:

$$\max ax, \text{ with constraint } \begin{array}{l} Ax \leqq \ell \\ x \geqq 0. \end{array}$$

From this, an abstract economy E can be derived in two ways: by introducing the sign constraint in either X or in Y (or in both).
First define $E := \{X,R,Y\}$ by:

$$X := \{x \in \mathbb{R}^{n} \mid ax \geq 0\}$$

$$R(x) := \{\tilde{x} \in X \mid a\tilde{x} \geq ax\}$$

$$Y := \{\tilde{x} \in X \mid Ax \leq \ell \text{ and } x \geqq 0\}.$$

Then as before:

---

1) See e.g. Gale (1968).

$$V^{\star} := \{p \in \mathbb{R}^{n\star} | p = \lambda a,\ \lambda \geqq 0\},$$

$$\hat{R}^{\star}_{+}(p) := \{\tilde{p} \in V^{\star} | \tilde{p} \geq p\}.$$

But now:

$$Y^{\star}_{-} := \text{Conv}\ \{a_1, a_2, \ldots, a_m\} + \mathbb{R}^n_-$$

$$= \text{Norm Conv}\ \{a_1, a_2, \ldots, a_m\}$$

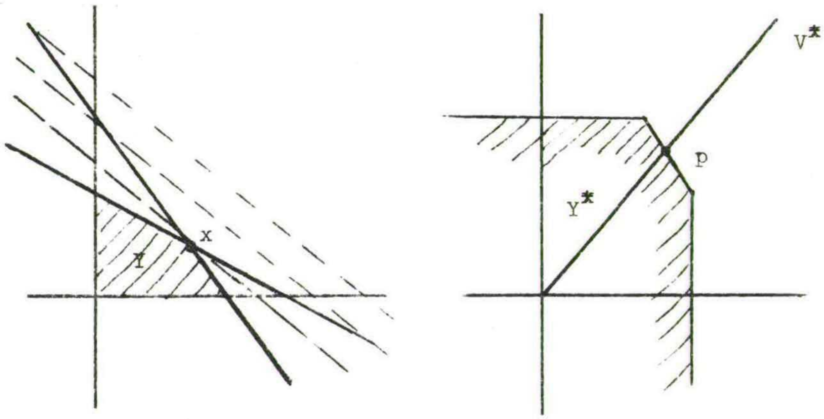

fig. 4.f.

For an optimum price in $E^{\star}$ we have again: $p = \lambda a$ such that:

$$\lambda a \in Y^{\star} \text{ and } \lambda' > \lambda \Rightarrow \lambda' a \notin Y^{\star}.$$

We now have:

$$Y^{\star} := \{p \in \mathbb{R}^{n\star} | \exists r \in \mathbb{R}^{m\star}:\ p \leqq rA,\ r \geqq 0,\ r\ell \leqq 1\}$$

(so $p = rA$ has been replaced by $p \leqq rA$).
So $p = \lambda a$ is optimal if:

$$\lambda = \max\ \{\lambda' \in \mathbb{R}_+ | \lambda a \leqq rA,\ r \geqq 0,\ r\ell \leqq 1\},$$

or equivalently:

$$\frac{1}{\lambda} = \min \{\mu \in \mathbb{R}_+ | a \leqq qA,\ q \geqq 0,\ q\ell \leq \mu\}.$$

This corresponds to the dual h.p problem:

$$\max q\ell$$
$$\text{s.t. } qA \geqq a$$
$$q \geqq 0.$$

Provided that $a > 0$, we could also define $\widetilde{E}^*$ by

$$V^* = \{p \in \mathbb{R}^n | p \geq \lambda a \text{ and } p \geqq 0\}$$

$$R_+^*(x) = \{\widetilde{p} \in V^* | \widetilde{p} \geq p\}$$

$$Y_-^* = \text{Conv}\ \{a_1, a_2, \ldots, a_m, 0\}$$

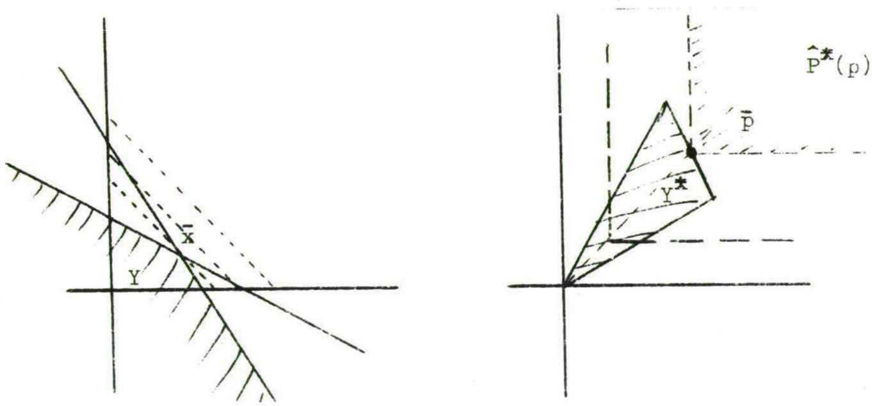

fig. 4.g.

Now an optimum price is $p \geq \lambda a$, $p \in X^*$ and

$$\lambda = \max \{\lambda \in \mathbb{R}_+ | \lambda a \leqq rA,\ r \geqq 0,\ r\ell \leqq 1\},$$

or:

$$\frac{1}{\lambda} = \min \{\mu \in \mathbb{R}_+ | a \leqq qA,\ r \geqq 0,\ r\ell \leqq \mu\},$$

which is identical to the corresponding formula above.
In E^* only prices colinear with a can be optimum; in $\tilde{E}^*$ all prices such that $p \geqq \lambda a$ may be optimum (but clearly not $p > \lambda a$).
The dual variable p in the model is the optimum price, whereas the prices q (or r) are the shadow prices of the restrictions.
Above we considered a maximum L.P. problem. A minimum L.P.-problem also has a dual, but then the preference correspondence in star shaped and the set of restrictions aureoled (min ax, given $Ax \geq b$, $b > 0$).
Particularly the dual program of the L.P. problem above (min $q\ell$, given $qA \geq a$, or $q'A \geq \ell$, for $a > 0$) is a minimum problem: clearly this problem has a dual optimum problem, that corresponds to the original L.P.-program.

5. Optimum and Nash Equilibrium.

5.1. An optimum formulated as a Nash equilibrium.

The abstract economy $E := \{X,P,Y\}$ with a single agent can be reformulated as an economy E_2 or E_3 with two or three agents. Under certain conditions, the optimum in E, in which the characteristic price vector remains implicit, corresponds to a Nash-equilibrium in E_2 or E_3, where the characteristic price appears explicitly.

(1) Three agents[1]:

$E_3 := \{X_i, P_i, Y_i, B_i\}$, for $i = 1,2,3$, where the action space becomes $R^{3n} = \mathbb{R}^{2n} \times \mathbb{R}^{n*}$, a typical element being (x,y,p):

agent 1:	$X_1 := X$
("consumer")	$P_1(x,y,p) := P(x)$
	$Y_1 := Y+N$, for $N = \{z \in \mathbb{R}^n \mid \lvert z\rvert < 1\}$
	$B_1(x,y,p) := \{z \in X_1 \cap Y_1 \mid pz \leq py\}$
agent 2:	$X_2 := \mathbb{R}^n$
("producer")	$P_2(x,y,p) := \{z \in X_2 \mid pz > py\}$
	$Y_2 := Y$
	$B_2(x,y,p) := Y_2$
agent 3:	$X_3 := R^{n*}$
("market")	$P_3(x,y,p) := \{p \in X_3 \mid p(x-y) > 0\}$
	$Y_3 := \{p \in X_3 \mid \lvert p\rvert \leq 1\}$
	$B_3(x,y,p) := Y_3$

1) A similar formulation for a different but related problem is given in Debreu (1952), see section 7.1.

Thus the single agent is split up into two agents, the first choosing a maximal element from $B_1(x,y,p)$, which will be called his <u>budget set</u>, the second chooses from $Y_2 := Y$ an element which maximizes the value at price p. A third agent, the "market agent", is added who chooses a price which maximizes the value of the difference between x and y.

<u>Theorem 5.1.</u> Let E and E_3 be economies in which X and Y are closed and convex, P has an open graph and is point-convex, $x \notin P(x)$ and $x \in \mathrm{Cl}\, P(x)$, then:

(a) if x is an optimum in E, $x \in \mathrm{Int}\, X$ and $p \neq 0$ is an optimum price, than $(x,y,\frac{p}{|p|})$ is a Nash-equilibrium in E_3 with $x = y$;

(b) if (x,y,p) is a Nash-equilibrium in E_3 than x is an optimum in E and $x = y$.

<u>Proof.</u>

(a) $H(p,px)$ separates Y and $P(x)$ and also $H(p,px)$ and $P(x)$.
Hence $P_1(x,y,\frac{p}{|p|}) \cap B_1(x,y,p/|p|) = \emptyset$ and $x \in B_1(x,x,p/|p|)$.
Also $P_2(x,y,p/|p|) \cap Y_2 = \emptyset$. Since $(x-y) = 0$, $P_3(x,y,p) \cap Y_3 = \emptyset$, $p/|p| \in Y_3$.

(b) By local non-satiation, $px = py$. If $(x-y) \neq \emptyset$, then for $q = \frac{x-y}{|x-y|}$, $q \in P_3(x,y,p) \cap Y_3$, which is impossible, hence $x-y = 0$.
Since $P_1(x,y,p) \cap B_1(x,y,p) = \emptyset$ and $P_2(x,y,p) \cap B_2(x,y,p) \neq \emptyset$, whereas $B_2(x,y,p) \supset Y_2 = Y$, $P(x) \cap Y = \emptyset$. □

<u>(2) Two agents:</u>

A simpler way to reformulate the optimum, is obtained by using duality; We have to assume now that:

$$0 \notin X \text{ and } 0 \in \mathrm{Int}\, Y$$

$E_2 := \{X_i,P_i,Y_i,B_i\}$, for $i = 1,2$, where the action space becomes $\mathbb{R}^n \times R^{n*}$, a typical element being (x,p).

agent 1: $X_1 := X$

$P_1(x,p) := P(x)$

$Y_1 := 2Y$

$B_1(x,p) := \{z \in X_1 \cap Y_1 \mid px \leq \sup pY\}$

agent 2: $X_2 := \mathbb{R}^{n*}$

$P_1(x,p) := \{q \in X_2 \mid qx > px\}$

$Y_2 := Y^*$

$B_2(x,p) := Y^*$

Thus the first agent chooses a maximal element from his budget set which now is based on maximization of the value on Y and agent 2 chooses a value maximizing price.

Theorem 5.2.: Given the assumptions of theorem 5.1.:

(a) if x is an optimum in E, $x \in \text{Int } X$ and p is an optimum price, than $(x,p/px)$ is a Nash equilibrium in E_2.

(b) If (x,p) is a Nash equilibrium in E_2, than x is an optimum with optimum price p in E.

Proof:

(a) $H(p,px)$ separates $P(x)$ and Y and $x \in \text{Bnd } Y$.
Since $x \in \text{Int } X$, $P(x) \cap H(p,px) = \emptyset$.
Hence also $P_1(x) \cap B_1(x,p/px) = \emptyset$. Since $H(p/px;1)$ supports Y in x, $H(x,1)$ supports Y^* in p/px. Hence $P_2(x,p/px) \cap Y^* \neq \emptyset$.

(b) Since $P_1(x,p) \cap B_1(x,p) = \emptyset$ and $B_1(x,p) \supset X \cap Y$, $P(x) \cap Y = \emptyset$.
Since $H(x,1)$ supports Y^* in p, $x \in Y$. □

5.2. A Nash-equilibrium reduced to an optimum.

An abstract economy $E_n := \{X_i, P_i, Y_i, B_i\}$ with n agents, can under certain conditions, be reformulated as an economy with a single agent E_1 [1)]

1) A similar result is given in Borglin and Keiding,(1976).A related model was considered in Ruys(1974).

Assume: $0 \notin X_i$ and $0 \in Y_i$ (for all i). Then $E_1 := \{X^H, P, Y^H, B^H\}$. The choice set, the planning set and the constraint correspondence are simply $X^H = \Pi X_i$ $Y^H = \Pi Y_i$ and $B^H(x^H) = \Pi B_i(x^H)$. The preferences $P_i : X^H \to X_i$ are aggregated into a single preference $P : X^H \to X^H$, by means of dual summation.
Let

$$\tilde{P}_i(x^H) := P_i(x^H) \times \prod_{j \neq i} X_j$$

and

$$L := \{\lambda \in \mathbb{R}^n \mid \Sigma \lambda_i = 1 \text{ and } \lambda \geqq 0\}$$

then:

$$P(x^H) := \bigcup_L \bigcap_i \lambda_i n \, \tilde{P}_i(x^H)$$

Theorem 5.3.: Let E_n and E_1 as defined above, be such that:

(1) $0 \notin X^H$, $0 \in Y^H$;

(2) for all i and $x^H \in X^H \cap Y^H$: $B_i(x^H)$ is closed and convex; and Star $B_i(x^H) \cap Y_i = B_i(x^H)$

(3) for all i, and $x^H \in X^H \cap Y^H$: $P_i(x^H) \neq \emptyset$, $x_i \in \mathrm{Cl}\, P_i(x^H)$ and $x_i \notin \mathrm{Conv}\, P_i(x^H)$.

Then the following statements are equivalent:

(a) x^H is a Nash equilibrium in E_n;

(b) x^H is an optimum in E_1.

Proof.

(a) Let $\bar{x}^H$ be a Nash-equilibrium in E_n, hence $x^H \in B(x^H)$.
Suppose $z^H \in P(x^H) \cap B(x^H)$, hence for all i, $z_i \in B_i(x^H)$.
Since for all i, $z_i \notin P_i(x^H) \cap B_i(x^H)$, $z^H \notin \tilde{P}_i(x^H)$; for $0 < \mu < 1$ and for all i : $\mu z_i \notin P_i(x^H)$: if $\mu z_i \in P_i(x^H)$, then $\mu z_1 \in$ Star $B_i(x^H) \cap P_i(x^H) = P_i(x^H) \cap B_i(x^H)$ and that is a contradiction. Hence $\mu z^H \notin \tilde{P}_i(x^H)$.
There must exist $\lambda \in L$, such that for all i, $z^H \in \lambda_i n \, \tilde{P}_i(x^H)$, but this is impossible, since for at least one i_0, $\lambda_{i_0} n \geq 1$, hence

$z^H \notin \lambda_{i_0} n \, \tilde{P}_{i_0}(x^H)$.

(b) Let x^H be an optimum in E_1, hence $x^H \in B(x^H)$ and $P(x^H) \cap B(x^H) = \emptyset$. Suppose for some $j \in I$, $P_j(x^H) \cap B_j(x^H) \neq \emptyset$, then for some $\lambda_j > \frac{1}{n}$, $\lambda_{jn} > 1$ and $\lambda_j n \, P_j(x^H) \cap B_j(x^H) = \lambda_j n \, P_j(x^H) \cap \text{Star } B_j(x^H) \neq \emptyset$, since $P_j(x^H)$ is open; let $z_j \in \lambda_j n \, P_j(x^H) \cap B_j(x^H)$.
By local non-satiation, $x_i \in \text{Cl } P_i(x^H)$ for all i.
Choose for all $i \neq j$, $\lambda_i = \frac{1-\lambda_j n}{n-1}$, hence $\lambda_i n < 1$ and therefore, by local non-satiation, for all $i, \lambda_i \, P_i(x^H) \cap B_i(x^H) \neq \emptyset$.
Let $z_i \in \lambda_i n \, P_i(x^H) \cap B_i(x^H)$. Hence $z \in B^H(x^H)$. Clearly $z \in \lambda_i n \, \tilde{P}_i(x^H)$ for all i. Hence $z \in \bigcap_i \lambda_i n \, \tilde{P}_i(x^H) \subset P(x^H)$ and that is a contradiction. □

6. Collective decisions.

As an application of the theory of the previous sections we consider a problem of collective decision making.
In section 2 we considered a single decision maker, in section 3 a group of decision makers, each optimizing independently, but under constraints dependent one other's choices. We have seen in section 5 that both problems are closely related.
In the present section we consider a group of decision makers who have to decide collectively on a single action. We shall see that this problem gives rise to optimization problems as defined in section 2.
A straight forward interpretation of the model is that of an economy with public goods only (see Ruys 1972, 1974).

Let the economy

$$\mathcal{E} := \{X_i, P_i, Y\}$$

be defined as follows:

$X_i \subset \mathbb{R}^n$ is a set of collective actions feasible for $i \in I$ (for I a (finite) set of agents), $P_i : X_i \to X_i$ a strict preference relation, and

$Y \subset \mathbb{R}^n$ is a constraint set.
Clearly the set $\cap X_i$ is the set of actions feasible for all agents.
$x \in \cap X_i$ may be interpreted as a vector of public goods, feasible for all consumers, Y as a set of producible public goods vectors.

6.1. Pareto optima in $\mathcal{E}$

A strong*) Pareto optimum in $\mathcal{E}$ is a solution $x \in \bigcap_i X_i \cap Y$, such that there exists no $y \in \bigcap_i X_i \cap Y$, where $y \in P_i(x)$ for all i.
A Pareto optimum in $\mathcal{E}$ is an optimum in the single agent abstract economy

$$E_1 := \{X, P, Y\}$$

for

$$X := \bigcap_i X_i$$

$$P(x) := \bigcap_i P_i(x)$$

i.e., in a Pareto optimum we have

$$x \in Y \text{ and } P(x) \cap Y = \emptyset.$$

From theorem 2.2 it follows that a Pareto optimum exists if $X \cap Y$ is compact and non empty, $P(x)$ has an open graph, is point convex and $x \notin P(x)$.
Clearly the conditions on P are fullfilled, if they hold for all P_i.
If for all i and x, also $x \in \mathrm{Cl}\, P_i(x)$, then $x \in \mathrm{Cl}\, P(x)$, if $P(x) \neq \emptyset$.
Let all the above conditions hold, assume also, that for all i there exists $\lambda > 1$, such that $\lambda x \in P_i(x)$. This implies that $P(x) \neq \emptyset$ for all x. for $x \in X \cap Y$ and $0 \in \mathrm{Int}\, Y$ and $0 \notin X$.

---

*) For a weak Pareto optimum it is required, that no y exists such that for all i, $x \notin P_i(y)$ and for some i, $y \in P_i(x)$. Under certain continuity condition on P_i, a weak Pareto optimum is also a strong Pareto optimum.

Then in a Pareto optimum, there exists a characteristic price p : $H(p,1)$ separates $P(x)$ and Y. Hence by property 4.9, $H(x,1)$ separates $R^*(x)$ and Y^* while $p \in P^*(x) \cap Y^*$.
By property 4.5 (6)

$$R^*(x) = \mathrm{Conv}(\bigcup_i P_i^*(x))$$

Hence $H(x,1)$ also separates all $P_i^*(x)$ from Y^*. By definition, $H(x,1)$ supports any $R_i^*(x)$.
There exists $p_i \in P_i^*(x)$ and $\alpha_i \geqq 0$, such that $\Sigma\, \alpha_i = 1$, for which

$$p = \Sigma\, \alpha_i p_i$$

We have $p_i x = 1$ and $H(p_i,1)$ supports $P_i(x)$ in x, for all i.
i.e. p_i is an optimum price for i, and p is some weighted mean of optimum prices. We could interpret this in the following ways:

(1) each consumer spends an amount 1, and fixes his shadow price p_i. The optimum price is a weighted mean of these prices;
(2) each consumer spends an amount α_i, $\bar{p}_i := \alpha_i p_i$ is his shadow price and the optimum price is the sum of these shadow prices.

6.2. Lindahl equilibrium.

A natural extension of the preceding model is to select among the Pareto optima a solution for which the weights α_i considered above, are fixed beforehand. This leads to a special case of a so called Lindahl equilibrium.*) Special in the sense that in the economy $\mathcal{E}$ only public (goods) actions are available.
The assumptions of the preceding section are retained.

It appears that the Lindahl equilibrium is an optimum in the abstract economy

$$E_1' := \{X, P', Y\}$$

*) See e.g. Milleron (1972), Ruys (1972).

derived from $\mathcal{E}_c$, where

$$X := \bigcap_i X_i$$

$$P'(x) := \bigcup_L \bigcap_i \frac{\lambda_i}{\alpha_i} P_i(x)$$

where $L := \{\lambda \in \mathbb{R}^H \mid \lambda_i \geqq 0 \text{ and } \Sigma\, \lambda_i = 1\}$.
Hence as a collective preference we use the dual sum (or convex intersection; def. 4.4.) of the individual preferences, which also appeared in theorem 5.2 in an essentially similar way).
It can be shown that P' has an open graph , is point-convex, $x \notin P(x)$ and $x \in \mathrm{Cl}\, P(x)$.

So in a Lindahl equilibrium we have

$$x \in Y \text{ and } P'(x) \cap Y = \emptyset$$

Its existence is guaranteed by the assumptions.
In the L.E. solution x, $H(p,1)$ separates $P'(x)$ and Y.
Clearly $H(x,1)$ separates $P'^{*}(x)$ and Y^{*}, whereas the intersection contains p.
By property 4.5. (8):

$$R'^{*}(x) = \Sigma\, \alpha_i\, R_i^{*}(x)$$

Hence if p is the optimum price, there exist $p_i \in R_i^{*}(x)$, such that $p = \Sigma\, \alpha_i p_i$.
Now $H(x,1)$ supports $R_i^{*}(x)$ in p_i and $H(p_i,1)$ supports $P_i(x)$ in x_1 for all i.
p is to be interpreted as a weighted mean of individual prices p_i, or as a sum of individual prices $\bar{p}_i := \alpha_i p_i$, where $\Sigma\, \bar{p}_i = p$ ($\bar{p}_i$ is called a Lindahl price), hence each individual spends α_i.
The solution $(x,p,(\bar{p}_i))$ is a Lindahl equilibrium (or public equilibrium) indeed, since it fulfils:

(1) for all i, x is the best in the set $\{x \mid \bar{p}_i x \leq \alpha_i\}$.

(2) $p = \Sigma \bar{p}_i$

(3) $py = \sup pY$.

In the case that the preferences of individuals are represented by a preordering $R_i(x)$, so that a dual preference relation $\hat{R}^*(p)$ exists, as considered in section 4.2, a Lindahl equilibrium may also be considered as a solution $(p,\bar{p}_i)$ such that

$$1.\ p \in \Sigma\ \alpha_i \hat{R}_i^*(p_i) \cap Y^*$$

$$2.\ \Sigma\ \alpha_i\ \hat{P}_i^*(p) \cap Y^* = \emptyset$$

while $H(x,1)$ separates $\Sigma\ \alpha_i\ \hat{R}_i^*(p_i)$ and Y.

Thus the Lindahl equilibrium in the price space becomes similar to an equilibrium in an economy with private goods as considered in the next section.

7. Competitive Equilibrium.

Let

$$\mathcal{E} := \{\{X_i\},\{P_i\},\ Y,\{w_i\},\{\theta_i\}\}$$

be an economy. H is a finite set of agents $H = \{1,2,\ldots,n\}$, there are n commodities. $X_i \subset \mathbb{R}^n$ is the consumption set of agent i, $P_i : X_i \to X_i$ is his preference correspondence and $w_i \in \mathbb{R}^n$ his resources. θ_i is i's share of total profits. $Y \ni 0$ is the production set. The income of agent i is

$$\varphi_i(p) := pw_i + \theta_i \sup pY$$

The budget set of agent i is

$$B_i(p) := \{x \mid px \leq \varphi_i(p)\}.$$

An <u>equilibrium</u> in $\mathcal{E}$ is an (n+1)-tuple $((x_i),p)$ such that:

(1) $\forall_i : P_i(x) \cap B_i(p) = \emptyset$

(2) $py = \sup pY$

(3) $y + \Sigma w_i = \Sigma x_i$

<u>7.1. An equilibrium in an economy formulated as a Nash-equilibrium.</u>

We can formulate an equilibrium in $\mathcal{E}$ as a Nash equilibrium in an abstract economy E_3 or E_2 derived from $\mathcal{E}$ in a way, similar to the method followed in section 3. (See e.g. Debren 1952, Sonnenschein Shafer 1975).

(1) by suitably formulating the behaviour of the producer and by adding a market agent (similar to (1) in section 5).

Define the abstract economy E_3 as follows:
The action space becomes $\mathbb{R}^{n(m+1)} \times \mathbb{R}^{n*}$, with as a typical element (x^H, y, p):

<u>agents</u> $i \in H$:

$$\hat{X}_i := X_i \subset \mathbb{R}^n$$

$$\hat{P}_i(x^H,y,p) := P_i(x_i)$$

$$\hat{Y}_i := 2(Y+\{w\})$$

$$\hat{B}_i(X^H,y,p) := \{x \in \hat{X}_i \cap \hat{Y}_i \mid pz \leq \varphi_i(p)\}$$

<u>agent</u> m+1:

$$X_{m+1} := \mathbb{R}^n$$

$$P_{m+1}(x^H,y,p) := \{z \in X_{m+1} \mid pz > py\}$$

$Y_{m+1} := Y$

$B_{m+1}(x^H, y, p) := Y$

agent m+2:

$X_{m+2} := \mathbb{R}^{n*}$

$P_{m+2}(x^H, y, p) := \{p \in X_{m+1} \mid p(\Sigma x_i - y) > 0\}$

$Y_{m+3} := \{p \in X_3 \mid |p| \leq 1\}$

$B_{m+3}(x^H, y, p) := Y_{m+3}$

Consumers are optimizing over their budget sets as required in (1) of the definition of an equilibrium. The producer maximizes profits and the (m+2)-nd agent, the market manager maximizes the value of the difference between consumption and production.
It can be shown that under suitable assumptions, the equilibrium in $\mathcal{E}$ corresponds to a Nash-equilibrium in the abstract economy E_3 (see Debreu 1952; Sonnenschein, Shafer 1975).
Note that we have in the Nash-equilibrium for $i \in H$: $H(p, \varphi_i)$ separates $P_i(x^H, y, p)$ and $B_i(x^H, y, p)$; $H(p, py)$ separates $P_{m+1}(x^H, y, p)$ and $B_i(x^H, y, p)$ whereas $P_{m+2}(x^H, y, p) = \emptyset$.

(2) by formalizing the producer as a price maker (as in (2) of section 5) in the abstract economy E_2.

The action space is $\mathbb{R}^{mn} \times \mathbb{R}^{n*}$, with a typical element (x^H, p).

agents $i \in H$:

$\hat{X}_i := X_i$

$\hat{P}_i(x^H, p) := P_i(x_i)$

$\hat{Y}_i := 2(Y + \{w\})$

$\hat{B}_i(x^H,p) := \{x \in \hat{X}_i \cap \hat{Y}_i \mid px \leq \varphi_i(p)\}$

agent m+1:

$\hat{X}_{m+1} := \mathbb{R}^{n*}$

$\hat{P}_{m+1}(X^H,p) := \{q \in X_{m+1} \mid q \Sigma x_i > p \Sigma x_i\}$

$\hat{Y}_{m+1} := (Y + \{w\})^*$

$\hat{B}_{m+1}(X^H,p) := (Y + \{w\})^*$

Again under suitable assumptions a Nash-equilibrium in the abstract economy E_2 corresponds to an equilibrium in $\mathcal{E}$.

7.2. An equilibrium characterized by an optimum price.

If the preferences in the economy $\mathcal{E}$ are given by a complete preordering, as considered in section 4.3, then we can characterize the equilibrium by an equilibrium price with which the equilibrium allocation can be associated. The equilibrium price can be formulated as an optimum in the sense of definition 2.1.
Assume that X_i and $R_i : X_i \to X_i$ fullfill the assumptions of section 7.1. Assume also that $\mathrm{Conv} \bigcup_i X_i \not\ni 0$. Let $Z := Y + \{w\}$ be starred, closed and convex.

Note first that an equilibrium in $\mathcal{E}$ is an (n+1)-tuple $((x_i),p)$, such that:

(1) $\Sigma P_i(x_i) \cap Z = \emptyset, \quad \Sigma x_i \in Z$

(2) $\forall_i : P_i(x_i) \cap B_i(p,\varphi_i(p)) = \emptyset, \; x_i \in B_1(p_1\varphi_i(p))$

for $B_i(p) = \{x \in X_i \mid px \leq \varphi_i(p)\}$.

Choose V_i such that $V_i = P_i(\bar{x}_i)$ for some $\bar{x}_i \in X_i$ and such that $V_i \cap Z = \emptyset$.
Let $\hat{P}^*_i : V^*_i \to V^*_i$ be as defined in section 4.3.
We define an aggregate preference in terms of prices: $\hat{P}^* : V^* \to V^*$ for $V^* = \cap V^*_i$.
Given $p \in V^*$, $\varphi_i(p)$ is i's income, hence $\frac{1}{\varphi_i(p)}$ is i's "personal" price, i.e. such that he may spend 1. $\hat{P}^*_i(\frac{1}{\varphi_i(p)}\, p)$ are the prices worse for i than $\frac{1}{\varphi_i(p)}\, p$. $\hat{P}^*$ is the dual sum of the individual preferences:

$$\hat{P}^*(p) = \bigcup_L \bigcap_i \hat{P}^*_i\left(\frac{1}{\varphi_i(p)}\right)$$

For P^* we have:

(1) $\hat{P}^*(p)$ is convex, aureoled;

(2) $\hat{P}^*$ has an open graph;

(3) $\hat{p} \notin \hat{P}^*(p)$ and $p \in \mathrm{Cl}\ \hat{P}^*(p)$.

The last property can be seen as follows:
for all i, $\frac{1}{\varphi_i(p)}\, p$ is on the lower boundary of $\hat{P}^*_i(\frac{1}{\varphi_1(p)}\, p)$; if we choose $\lambda_i = \varphi_i(p)$, them for all i

$$p \in \mathrm{Bnd}\ \varphi_i\ \hat{P}^*_i\ \left(\frac{1}{\varphi_i(p)}\right)$$

if $\lambda_i \neq \varphi_i(p)$, for some i, then for some j: $\lambda_j > \varphi_j(p)$.
Then $\frac{\lambda_j}{\varphi_j(p)}\, p \in \mathrm{Bnd}\ \lambda_j\ \hat{P}^*_j\ (\frac{1}{\varphi_j(p)}\, p)$, but since $\frac{\lambda}{\varphi_i(p)} > 1$, $p \notin \lambda_j \hat{P}^*_j(\frac{1}{\varphi_j(p)}\, p)$.
This proves (3) above.

Let p be an optimum w.r.t. $\hat{P}^*$ in Z^*:

$$p \in Z^* \text{ and } \hat{P}^*(p) \cap Z^* = \emptyset.$$

Then $H(z)$ separates $\hat{P}^*(p)$ and Z^* and z is a "characteristic action" at p in E^{*}. From property 4.9:

$$\text{Rint } \hat{P}^{**}(p) \cap Z = \emptyset \text{ and } x \in Z.$$

From property 4.5(8):

$$\hat{P}^{**}(p) = \Sigma\, \hat{P}_i^{**}\left(\frac{1}{\varphi_i(p)}\, p\right)$$

Hence there exist $x_i \in \text{Bnd } \hat{P}_i^{**}\left(\frac{1}{\varphi_i(p)}\, p\right)$, such that $\Sigma\, x_i = x$.
We have $\frac{1}{\varphi_i(p)}\, p\, x_i = 1$, and $x_i' \in \hat{P}_i^{**}\left(\frac{1}{\varphi_i(p)}\, p\right)$ implies $\frac{1}{\varphi_i(p)}\, p\, x_i' > 1$.
Hence $p\, x_i = \varphi_i(p)$ and x_i is best in the budgetset.
So $((x_i),p)$ is an equilibrium.

8. Dynamic technological economies.

One way to introduce time into the models presented above is to add a number of dimensions to the spaces X and Y and to label these dimensions by a sequence of time period. The result is, in fact, a reinterpretation of the models above. It does not help the decision makers much, because they must have (perfect) knowledge of all actions and prices in future, while it is difficult enough to require knowledge of present actions and prices.
Economists have therefore tried to give some structure to future actions, but have focussed their attention mainly to the production side of the economy, considering consumption as a necessary input to production.
In this section, a generalization of the Neumann-Gale model of production will be given.
In next section, a structure in time on consumption wil be developed, and the theory presented above will be applied.

8.1. A dynamic technological economy.

Definition 8.1.

Let the economy $\mathcal{E} := \{T,(K_t),(P_t),(Y_{t\tau}),x_0\}$ be defined by:[1)]

(1) a finite set $T := \{0,1,2,\dots,T\}$ indicating time periods;

(2) a class $(K_t)_{t\in T}$ of nonnegative orthants, $K_t := \mathbb{R}_+^{n_t}$, indicating possible actions at t;

(3) a class $(P_t)_{t\in T}$ of preference preorderings on K_t, where P_T is monotonous and convex, and P_t, $t = 0,1,\dots,T-1$, is defined by $P_t(x) := \text{Int}\,[\{x\} + K_t]$;

(4) a class $(Y_{t,\tau})_{t,\tau\in T}$ of superlinear and monotonous correspondences from K_τ into K_t, $\tau < t$, such that $Y_{t,t-s} \circ Y_{t-s,\tau} = Y_{t,\tau}$, for $t > t-s > \tau$ indicating the producer's "law of motion", i.e. the set of actions feasible at t, given an input at τ;

(5) the initial resources x_0 at $t = 0$.

This economy $\mathcal{E}$ is said to be a dynamic <u>technological economy.</u>

The states (x_t) of the economy over time give a description of the motion of the economic system. A set of states $(x_t)_{t\in T} =: x^T$ in is called a <u>trajectory</u> of $\mathcal{E}$, if:

(1) $x_t \in K_t$, for $t \in T$;

(2) $x_t \in Y_{t,\tau}(x_\tau)$, for $t,\tau \in T$ and $t > \tau$;

(3) $x_0 = x_t$, for $t = 0$.

Definition 8.2.

The set of trajectories in $\mathcal{E}$ is denoted by $\mathcal{X}$.

A trajectorie $x^T \in \mathcal{X}$ is said to be P_T-<u>optimal</u> if:

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*) Many results can also be obtained for more general economies (see Makarov and Rubinov, 1977). The proofs of the properties in this section can be found in the book of Makarov and Rubinov, Some concepts however, viz. optimality and duality, are adapted here to the problem under consideration.

1) $x_T \in K_T \cap Y_{T,0}(x_0)$,
2) $P_T(x_T) \cap Y_{T,0}(x_0) = \emptyset$.

Each of the following conditions in $\mathcal{E}$ is necessary and sufficient condition for a trajectory x^T to be P_T-optimal:

a) x_T is an upper-boundary element of Norm $Y_{T,0}(x_0)$;

b) x_0 is a lower-boundary element of Mon $Y^{-1}_{T,0}(x_T)$;

c) there exists a non-zero price $P_T \in K^*_T := (\mathbb{R}^{n_T}_+)^0_+$ such that $p_T x_T = \max\{p_T x \mid x \in Y_{T,0}(x_0)\} > 0$;

d) there exists a nonzero price $p_0 \in K^*_0$ such that $p_0 x_0 = \inf\{p_0 x \mid x \in Y^{-1}_{T,0}(x_T)\} > 0$.

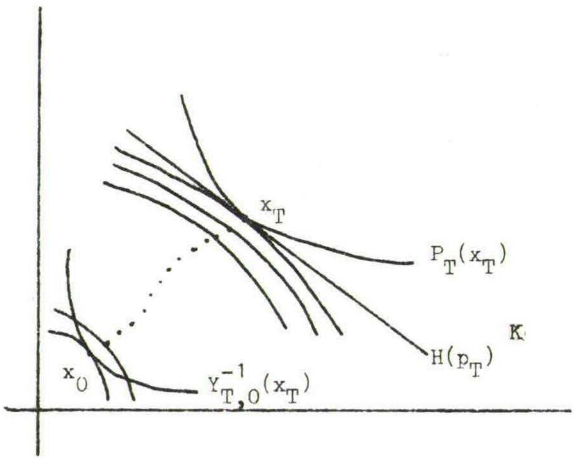

fig. 8.a.: Optimal trajectory, for $K_t = \mathbb{R}^2_+$

Although the criterion of optimality in definition 8.2 has been placed at the final stage T, it implies efficiency at all preceding time periods. This is expressed by the following property, called the first principle of optimality:

If x^T is a P_T-optimal trajectory of $\mathcal{E}$, than for any $t \in T$ the trajectory x^t in $\mathcal{E}_t$, with $T := \{0,1,\ldots,t\}$, is a P_t-optimal trajectory of $\mathcal{E}_t$. So, any t-section of a P_T-optimal trajectory is P_t-optimal in restricted economy $\mathcal{E}_t$ of $\mathcal{E}$. This principle may also be called the efficiency or value conserving property of an optimal trajectory, because at each stage between 0 and T, no action is decided upon that is inefficiënt or "looses" a positive amount of resources available at each stage.

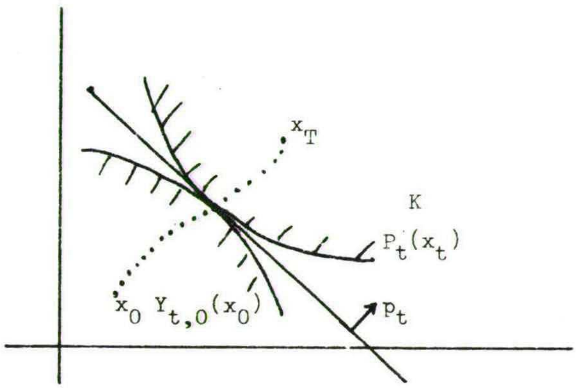

fig. 8.b.

An abstract economy $E_1 := \{X,P,Y,x_0\}$ can be defined that corresponds with $\mathcal{E}$:

$$X := K_1 \times K_2 \times \ldots \times K_T,$$
$$P := P_1 \times P_2 \times \ldots \times P_T,$$
$$Y := \{x_0\} \times Y_{1,0}(x_0) \times \ldots \times Y_{T,0}(x_0).$$

Since $X \cap Y$ is compact, convex and nonempty; P has an open graph, $x \notin \text{Conv } P(x)$, and $x \in \text{Cl } P(x)$, for all x, it follows from proposition 2.2 that there exists an optimum in E_1, i.e.:

$$x \in X \cap Y \text{ and } P(x) \cap Y = \emptyset$$

This implies that there exists a P_T-optimal trajectory x^T in $\mathcal{E}$.
It is easily verified that this trajectory is also a Nash-equilibrium in $\mathcal{E}$, interpreting the agents in the abstract economy E_H by time periods (see definition 3.1): $= E_T$.
The concepts of Nash-equilibrium, Pareto-optimality and the principle of optimality mentioned above, coincide in the economy $\mathcal{E}$.
A dual model of the economy $\mathcal{E}$,

$$\mathcal{E}^* := \{T, (K_t^*), (P_t^*), (Y_{t\tau}^*), p_T\}$$

is defined by:

(a) the set T of time periods in $\mathcal{E}$;

(b) for each t, $K_t^* := (\mathbb{R}_+^{n_t})_+^0$, indicating possible prices;

(c) for each $t \neq T$, $P_t^*(p) := \text{Int}\,[\{p\} + K_t^*]$, and $P_T^*(p) := P_T^*(p)$, as defined in section 4.3;

(d) $Y_{t,\tau}^* := Y_{t,\tau}^{\circledast}$, as defined in section 4.2, with $\pi \equiv 0$;

(e) $p_T \in [Y_{T,0}(x_0)]^*$.

Since all the conditions in the definition of the dynamical technological economy $\mathcal{E}$ are met, the model $\mathcal{E}^*$ is called a <u>dual technological economy.</u>

In this "economy" a set of states (p_t) describes the motion of the system over time. A trajectory p^T has to be feasible, as defined above. A trajectory p^T is also P_T^*-optimal, if it meets the conditions of definition 8.1.

Since the graph of $Y_{t,\tau}^*$ is a cone, for each $t,\tau \in T$ and $p_1 \in Y_{1,0}^*(p_0)$, $p_2 \in Y_{2,1}^*(p_1), \ldots, p_T \in Y_{T,T-1}^*(p_{T-1})$, it follows from the duality operation $\circledast$ that:

(1) $$p_0x_0 \geqq p_1x_1 \geqq \cdots \geqq p_{T-1}x_{T-1} \geqq p_Tx_T,$$

for any trajectory $x^T \in X$.

Further, if trajectory p^T is P_T^*-optimal, i.e. $p^T \in Y_{T,0}^*(p_0)$ and $Y_{T,0}^*(p_0) \cap P_T^*(p_T) = \emptyset$, and if trajectory x^T is P_T-optimal, then:

(2) $\quad p_T x_T = p_{T-1} x_{T-1} = \dots = p_0 x_0.$

Finally, from monotonicity of P, and $x_0 \neq 0$, follows:

(3) $\quad p_T x_T > 0.$

The P_T^*-optimal trajectory p^T is called a <u>characteristic trajectory</u> of a P_T-optimal trajectory x^T.

Since the existence of a characteristic trajectory is guaranteed, condition (2) above expresses the second principle of optimality: (value equalizing over time): for any optimal trajectory x^T in $\mathcal{E}$, there exists a characteristic trajectory p^T in $\mathcal{E}^*$ such that each action x_t has a constant value.
This is also expressed in the duality relation:

$$\max p_T [Y_{T,0}(x_0)] = \inf [Y_{T,0}^{*-1}(p_T)] x_0$$

It may be noticed that any P_T^*-optimal trajectory p^T consists of a set of Nash-equilibrium prices.

<u>8.2. A Neuman-Gale economy.</u>

Let a dynamic technological economy $\mathcal{E}$ be such that:

(1) $K_t = K = \mathbb{R}_+^n$, for all $t \in T$;

(2) $Y_{t+1,t} = Y$, for $t = 0,1,\dots,T-1$.

This economy is called a Neuman-Gale economy. The problem to be analysed is the rate of growth of this economy, irrespective of the initial or terminal action x_0, resp. x_T.

We therefore restrict out attention to an arbitrary process $(x,y) \in Gr(Y)$, relating input x and output y between two consecutive time periods.

An equilibrium state in $\mathcal{E}$ consists of a positive number α, a process $(x,y) \in Gr(Y)$ and a price $p \in K^*$ such that:

a) $\alpha x \leqq y$;

b) $\alpha p\tilde{x} \geqq p\tilde{y}$, for all $(\tilde{x},\tilde{y}) \in Gr(Y)$;

c) $py > 0$.

The scalar α is called a rate of growth for $\mathcal{E}$. It is not necessarily the maximum rate of growth. This is called the technological or Neumann rate of growth, and defined by:

$$\alpha(Y) := \max \{\alpha \mid \exists (x,y) \in Gr(Y), \|(x,y)\| \leqq 1: \alpha x \leqq y\}.$$

A process, resp. an equilibrium state, that has a Neumann rate of growth, is called Neumann; and a trajectory consisting of such processes is called a turnpike, or a Neumann-ray. It has been shown that P_t-optimal trajectories can only temporarily deviate from a neighborhood of a turnpike, if they do so anyway.[1)]

An abstract economy $E_1 := (X,P,B)$ is defined by

$$X := K \times K$$
$$P(x,y) := \{(\tilde{x},\tilde{y}) \mid \exists \alpha : \alpha x = y \text{ and } \alpha\tilde{x} < \tilde{y}\};$$
$$B := Gr(Y) \cap \{(x,y) \mid \|(x,y)\| \leqq 1\}.$$

Since the conditions for the existence of an optimum are satisfied, there exists an $(x,y) \in X \cap B : P(x,y) \cap B = \emptyset$.
This optimum satisfies condition (a) of an equilibrium state, and implies a Neumann rate of growth.

---

1) A recent summary of developments in turnpike theory can be found in McKenzie (1976).

Further, condition (b) is equivalent to saying that $\frac{1}{\alpha} p \in Y^{*}(p)$, and indicates the "rate of growth" of the dual model. If $x \neq 0$, condition (c) is also met and thus existence of an equilibrium state has been shown.

The economic rate of growth of $\mathcal{E}$ is defined by:

$$\beta := \min_{p \geq 0} \max_{y \in Y(x)} \frac{py}{px}.$$

At a Neumann equilibrium state, $\beta = \alpha$. The actions are valued at the same prices, p. If this proces (x,y) are two consecutive states in an optimal trajectory, and prices belong to the characteristic trajectory, then

$$px = qy = (\frac{1}{\alpha} p)(\alpha x)$$

Example: The von Neumann model is an economy $\mathcal{E}$ in which the thechnology is defined by the following production correspondence $Y : \mathbb{R}^n_+ \to \mathbb{R}^n_+$:

$$Y(x) := \{y \mid \exists\ z \in \mathbb{R}^m_+ : y = Bz \text{ and } x = Az\}.$$

Its dual model is $Y^* : \mathbb{R}^{n*}_+ \to \mathbb{R}^{n*}_+$ defined by:

$$Y^*(p) := \{q \mid pA \geqq qB\}.$$

8.3. A convex technological economy.

An economy $\mathcal{E} := \{T,(K_t),(P_t),(Y_{t\tau}),x_0\}$ is called a convex technological economy if it meets all conditions of a technological economy defined in 8.1, except for $Y_{t,\tau}$ being convex star correspondences rather than superlinear maps.

The concepts in $\mathcal{E}$, defined in 8.1, remain unchanged.
The definition of a dual economy $\mathcal{E}^*$, however, needs some adstruction, because the consistency condition (4) in definition 8.1 requires that consecutive prices are interdependent and therefore also the scalars in the duality operation.

A dual model of the economy $\mathcal{E}$ is defined by:

$$\mathcal{E}^* := \{T,(K_t^*),(P_t^*),(Y_{t\tau}^*),p_T\}$$

Since $Y_{t\tau}$ is a convex star correspondence, $Y_{t\tau}^*$ is defined by $Y_{t,\tau}^{\circledast} : \mathbb{R}^{n*} \times \mathbb{R} \to \mathbb{R}^m$. The consistency condition of the composition operation requires that the scalar's values satisfy: $\pi_{t,t-s} + \pi_{t-s,\tau} = \pi_{t,s}$, for $t > t-s > \tau \geqq 0$.

Let $\pi_{t,s} := \sum_{i=s}^{t} \pi_i$, and $\sum_{i=0}^{T} \pi_i = 1$.

Then $\forall x_\tau$, $\forall x_t \in Y_{t\tau}(x_\tau)$: $p_\tau x_\tau + \pi_{t,\tau} \geqq p_t x_t$, with $0 \leqq \tau < t \leqq T$.
In this "economy" $\mathcal{E}^*$, the motion of the system over time is described by a set of states (p_t,π_t). Such a sequence (p^T,π^T) is called a <u>trajectory</u> in $\mathcal{E}^*$, if: $p_t \in Y_{t,\tau}^{\circledast}(p_\tau,\pi_{t,\tau})$, and $\pi_{T,0} = 1$, $\pi_T = 0$.

The sequence π^T is decreasing and $\pi_T = 0$. Each π_t may be interpreted as the <u>profit</u> over period t.
Equivalently, one can use scalars $\mu_t := \pi_t + \mu_{t+1}$ to indicate the relation of p^T over time.[1)] If p^T is a trajectory, then:

(1) $$p_0x_0 + \mu_0 \geqq p_1x_1 + \mu_1 \geqq \cdots \geqq p_{T-1}x_{T-1} + \mu_{T-1} \geqq p_Tx_T.$$

Since $\sum_{s=0}^{T} \pi_s = 1$, it follows that $\mu_0 = 1, \mu_{T-1} = \pi_{T-1}, \mu_T = \pi_T = 0$, and $\mu_t = 1 - \sum_{s=0}^{t-1} \pi_s$. Further, if p^T is P_T^*-optimal, and x^T is P_T-optimal then:

(2) $$p_0x_0 + 1 = p_1x_1 + \mu_1 = \ldots = p_{T-1}x_{T-1} + \mu_{T-1} = p_Tx_T$$

If also

(3) $$p_Tx_T > 0,$$

---

1) This approach is followed by Makarov and Rubinov.

then (p^T, π^T) is called a characteristic trajectory of a p_T-optimal trajectory x^T. The abstract economy must now be extended for a set of "agents", because a decision at one time period t about the scalar π_t influences the actions at other time periods.

Let $E_T := (X_t, P_t, Y_t, B_t)$ be defined by:

$$X_t := K_t^0 \times [0,1],$$

$$P_t(p_t, \pi_t) := \prod_{t=0}^{T} \operatorname{Int}\big([\{p_t\} + K_t^0] \times [\{\pi_t\} + \mathbb{R}_+]\big),$$

$$Y_t := X_t,$$

$$B_t(p_t, \pi_t) := \prod_{t=0}^{T} \left(Y_{t-1,t}^{*-1}(p_t, \pi_t) \times [0, \pi_t]\right), \text{ with } \Sigma\pi_t = 1.$$

A Nash-equilibrium is a trajectory (p^T, π^T) which is feasible and optimal for each period with respect to the simple preference relation used here (see definition 3.1).

If T is finite, this Nash-equilibrium can be shown to exist. The Nash-prices $(x_t, 1)_{t \in T}$ of this equilibrium in $\mathcal{E}^*$ generate hyperplanes $H(x_t, 1)$ such that $P_t^*(p_t^T, \pi_t^T)$ and $Y_t^*(p^T, \pi^T)$ are separated.

The second principle of duality (value equalizing) is also expressed by the duality relation:

$$\max p_T[Y_{T,0}(x_0)] = \inf [Y_{T,0}^{*-1}(p_T, 0)]\ x_0 + 1.$$

The interesting feature of a characteristic trajectory in a dual convex technological economy is, of course, the explicit distribution of profits over time, π^T. This distribution has been made explicit by the assumption of marginal pricing, which can be implied institutionally by a regime of, e.g., perfect competition.

9. Dynamic consumption economies.

The technological models introduced in the previous section have a well developed production structure, but only a quite rudimentary consumption structure. For consumption is only admitted in the last period if the composition of the bundle x_T is based on consumer's preferences. All intermediate stages are producer's business.
The general models of the other sections permit interpretations conforming which at the initial stage consumer decisions are made for all future periods. This solution to the problem is not more attractive, because it implies strong assumptions about foresight and institutions.

Another approach is to consider the consumption sector as one of the production sectors, producing "utility" from an input bundle. A trajectory is called optimal if it cannot be overtaken by another trajectory with a higher sum of utility over time. This problem can be formalized into a convex technological economy (see e.g. Makarov and Rubinov, 1977). In this paper, however, consumption will be treated differently.

First, a consumption pattern over time will be introduced that is based on an aspiration process over time, based on actual consumption at present (and in de past). This structure is simpler than a complete preference preordering over all trajectories, as is usually assumed to solve the problem of intertemporal allocation. It also allows for a distinction between immediate consumption and future consumption, the latter generating a "preference" relation for demand of resources.

Let $K^T := K_0 \times K_1 \times \ldots \times K_T$ be the action or consumption space and $c^T \in K^T$ a consumption trajectory.

The usual criterion for optimality of c^T is given by the discounted sum of utility over time. Let $u_t(c_t)$ be the utility of c_t at stage t, and ψ^{-t} the discount factor, then:

$$P(\bar{c}^T) := \{c^T \mid \Sigma\, \psi^{-t} u_t(\bar{c}_t) < \Sigma\, \psi^{-t} u_t(c_t)\}.$$

This definition implies a complete preordering on K^T, and allows for substitution of consumption between time periods (see fig. 9.a).

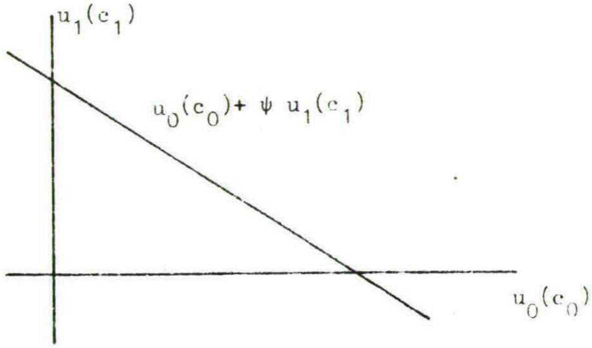

fig. 9.a.

Intertemporal utility comparison is also implied by this criterion, as total utility from c^T is:

$$u(c^T) := \Sigma\, \Psi^{-t}\, u_t(c_t).$$

Instead of this criterion, we will assume that for each time period the consumer's welfare depends on some <u>level of aspiration</u> ψ_t based on initial or actual consumption.

<u>Definition 9.1.</u>
Let K_τ and K_t be the consumption spaces, with $\tau < t$, and P_τ a monotonous, convex preference relation on K_τ. The function $\psi_{t\tau} : K_\tau \rightarrow K_t$ assigning to each consumption bundle c_τ a bundle c_t, such that:

1) $[\hat{c}_\tau \in \text{Bnd } P_\tau(c_t)] \Rightarrow [P_t(\psi_{t\tau}(\hat{c}_\tau)) = P_t(\psi_{t\tau}(c_\tau))]$,

2) $\psi_{t,t-s} \circ \psi_{t-s,\tau} = \psi_{t,\tau}$,

3) $\psi_{t,\tau}$ is continuous and one-to-one,

is said to be an <u>aspiration function.</u>
The preference relation $P_{t\tau}$ on K_t, generated by a given preference relation P_τ on K_τ and the aspiration function $\psi_{t\tau}$ is called an <u>induced preference relation</u> and defined by:

$$P_t(c_t) := \{\psi_{t\tau}(c_\tau) \mid c_\tau \in P_\tau(\psi_{t\tau}^{-1}(c_t))\}.$$

The preference relation at t indiced by P_0 will be denoted without second suffix: $P_{t0} =: P_t$, and $\psi_{t0} =: \psi_t$.

If the consumption spaces have equal dimension, then the preference relations at a certain stage can be derived from:

$$P_t(c) := \Psi_t(P_0(c)),$$

which is equal to $P_0(\psi_t(c))$ if $K_t = K_0$.

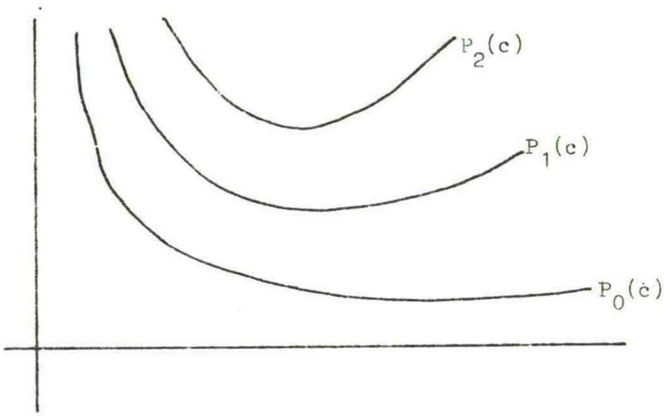

fig. 9.b. Induced preference relations over time.

The intertemporal preference relation P is based on the idea of increasing or maintaining an aspiration consumption level over time. This implies that total utility derived from a consumption trajectory is determined by the lowest utility at any stage in the trajectory. Attention is thus focussed on bottlenecks in the trajectory (maximizing minimal stages). This can be achieved as follows. Let (x_0, c_1) be a consumption trajectory in $K_0 \times K_1$. The preference relation on $K_0 \times K_1$ is defined by:

$$P(c_0,c_1) := [P_0(c_0) \times P_1(\psi_1(c_0))] \cup [P_0(\psi_1^{-1}(c_1)) \times P_1(c_1)].$$

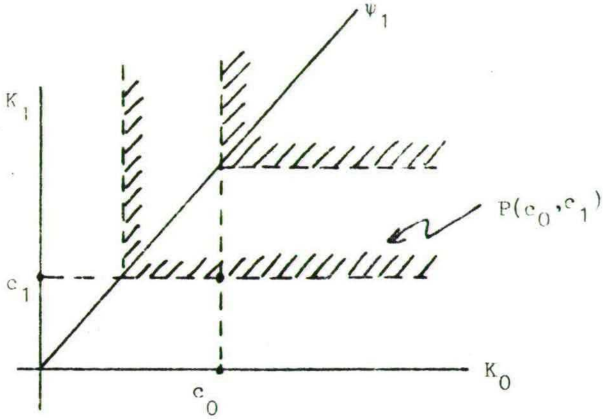

fig. 9.c. The preference relation on $K_0 \times K_1 = \mathbb{R}_+ \times \mathbb{R}_+$.

In general, the preference relation is given by:

Definition 9.2.
Let $c^T \in K^T$ be a consumption trajectory. Then the intertemporal preference relation is defined by:

$$P(c^T) := [P_0(c_0) \times P_1(\psi_1 c_0) \times \ldots] \cup [P_0(\psi_1^{-1} c_1) \times P_1(c_1) \times \ldots] \cup \ldots$$

$$\ldots \cup [P_0(\psi_T^{-1} c_T) \times P_1(\psi_{T-1}^{-1} c_T) \times \ldots]$$

$$= \bigcup_{i=0}^{T} \prod_{t=0}^{T} P_t(\psi_{t-i}(c_i)), \text{ with } \psi_{t-i} := \psi_{i-t}^{-1}, \text{ if } t < i.$$

This preference relation on K^T can be reduced to a correspondence $P_0 : K^T \to K_0$, because of the special structure of preferences generated by the aspiration function.

Definition 9.3. The intertemporal preference correspondence $P_0 : K^T \to K_0$ is defined by:

$$P_0(c^T) := P_0(c_0) \cup P_0(\ {}_1^{-1}(c_1)) \cup \ldots \cup P_0({}_T^{-1}(c_T)).$$

$$= \bigcup_{i=0}^{T} P_0(\psi_i^{-1}(c_i)), \text{ with } \psi_0 = 1.$$

If, for example, the aspiration function $\psi_{t,\tau}$ is a scalar such that $\psi_{t,0} =: \psi^t$, and a utility function u can be defined on each K_t, then the optimality criterion for c^T in K^T is given by the minimal utility of any time period, discounted by ψ^t:

$$P(\bar{c}^T) := \{c^T \mid \min_{t\in T} \{\psi^{-t}u(\bar{c}_t)\} < \min_{t\in T} \{\psi^{-t}\ u(c_t)\}\}.$$

Contrary to the usual criterion of discounted summation of utilities over time, this criterion gives only a partial preordering on K^T, and requires intertemporal complementarity rather than substitution between time periods.

We are now able to define a <u>dynamic superlinear consumption economy.</u>

$$\mathcal{E} := \{T,(K_t),P,(Y_{t\tau}),y_0\},$$

by:
a finite set T of time periods and a set (K_t) of nonnegative orthants; an intertemporal preference relation P, which is generated by a monotonous and convex preference relation P_0 on K_0 and a set of aspiration functions $\psi_{t,\tau}$; a set of monotonous, point-convex and superlinear production correspondenced $(Y_{t,\tau})$ and a vector of initial resources y_0.

Both $(\psi_{t,\tau})$ and $(Y_{t,\tau})$ have to satisfy the compatibility condition with respect to the operation of composition.
The feasibility condition on a consumption trajectory c^T in is rather complicated, as it depends on the technology and the resources invested for future consumption.

A <u>trajectory</u> (c^T,x^T) in $\mathcal{E}$ is <u>feasible</u> if:

(1) $\forall t \in T : y_t := (c_t+x_t) \in Y_{t,t-1}(x_{t-1});$

(2) $c_0 + x_0 = y_0$, and $c_T = y_T$.

This implies that a consumption trajectory c^T is feasible if and only if:

$$c_T \in Y_{T-1,T-2}(\dots Y_{2,1}(Y_{1,0}(y_0-c_0)-c_1) \dots - c_{T-1}).$$

Therefore, with each $y_0 \in K_0$ a set of feasible consumption trajectories can be associated. This set will be denoted by $\mathfrak{C}(y_0)$.

Definition 9.4.
A consumption trajectory c^T in $\mathcal{E}$ is said to be optimal if:

1) $c^T \in K^T \cap \mathfrak{C}(y_0)$, and
2) $P(c^T) \cap \mathfrak{C}(y_0) = \emptyset$.

We now can define an abstract economy:

$$E_1 := \{K^T, P, \mathfrak{C}(y_0)\},$$

which represents the economy $\mathcal{E}$ through the concepts K^T, P and (y_0) defined above. Since the conditions of proposition 2.2 are satisfied, there exists an optimum c^T.

Next, it will be shown that the special preference and production structure of the economy $\mathcal{E}$ permits us to reduce the time dimension of the problem to one and to formulate the decision problem in the initial stage.

If $c^T \in \mathfrak{C}(y_0)$, then it is known which bundle should be invested at t=0 to provide the future consumption: $x_0 = y_0 - c_0$. If, however, no decision is yet made about actual consumption, but future consumption is given by the future consumption trajectory $c_0^T := (c_1, c_2, \dots, c_T)$, then the investment x_0 at t = 0 to provide for this future consumption must be an element of the set

$$I(c_{-0}^T) := \{x_0 \in K_0 \mid x_0 = y_0 - c_0 \text{ and } c^T \in \mathfrak{C}(y_0)\}.$$

$$= \{x_0 \in K_0 \mid x_0 \in Y^{-1}_{1,0}(c_1 + Y^{-1}_{2,1}(c_2 + \ldots + Y^{-1}_{T,T-1}(c_T)))\}.$$

Since this correspondence $I : K^T_{-0} \to K_0$ associates with a future consumption trajectory c^T_{-0} the set if resources x_0 at least necessary to provide for this future trajectory, given the production technology, I is said to be the investment correspondence at t=0. This correspondence is point-convex, point aureoled and monotone in K_0, since $Y_{t,\tau}$ is a superlinear, point-starred and normal map, for each $t,\tau \in T$ (see fig. 9.d).

Using the preference correspondence from definition 9.3, the optimality criterion of definition 9.4 can be placed in K_0. For, a consumption trajectory c^T is optimal if and only if:

1) $c_0 \in K_0 \cap B_0(c^T) := \{c \in K_0 \mid c \in \{y_0\} - I(c^T_{-0})\}$,

2) $P_0(c^T) \cap B_0(c^T) = \emptyset$.

But also c^T can be reduced to K_0, as a result from the special preference structure.

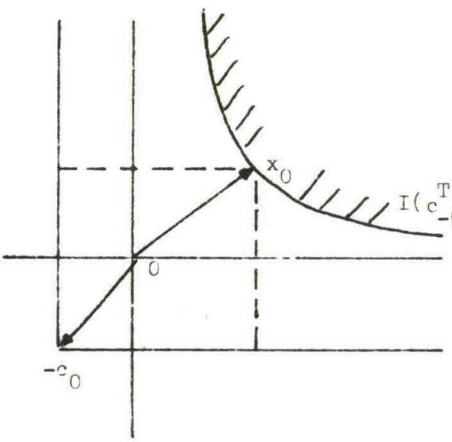

fig. 9.d.

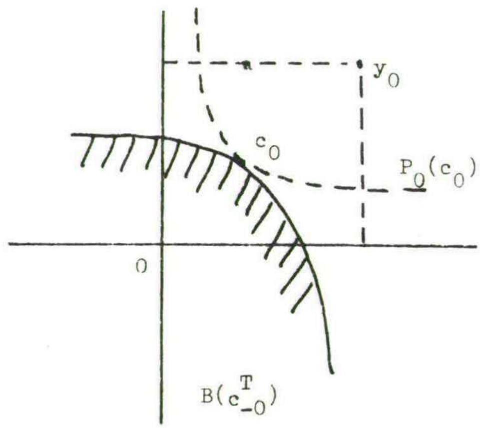

fig. 9 e.

Thus, given some future consumption trajectory c^T_{-0}, and the initial resources, the choice of consumption at the initial stage, c_0, must be made from the choice-set

$$B(c^T_{-0}) := \{y_0\} - I(c^T_{-0})$$

The dependence of future consumption, however, on initial or actual consumption is expressed in definition 9.1. Since P_0 is monotonous, Cl P = R, we define:

$$R^T_{-0}(c_0) := \{c^T_{-0} \mid \prod_{t=1}^{T} R_t(c_0)\} = \{c^T_{-0} \mid \prod_{t=1}^{T} \psi_{t,0}(R_0(c_0))\},$$

and

$$B_0(c_0) := [\{y_0\} - I(R^T_{-0}(c_0))] \cap K_0.$$

Thus we have derived from $\mathcal{E}_T$ an abstract <u>economy E_0 in the initial stage:</u>

$$E_0 := \{K_0, P_0, Y_0, B_0\},$$

with $Y_0 := \{x \in K_0 \mid x \leqq y_0\}$, the planning set of E_0, and B_0 as defined above: the constraint correspondence from $K_0 \cap Y_0$ into itself, containing all information about future consumption aspirations and future production technology of model $\mathcal{E}$.

Again, since the conditions of proposition 2.2 and definition 2.4 are satisfied, there exists an optimum c_0, i.e.

$$c_0 \in K_0 \cap B_0(c_0), \text{ and}$$

$$P_0(c_0) \cap B_0(c_0) = \emptyset$$

From the construction of E_1 and E_0 follows:

<u>Proposition 9.5.</u> Let c^T be an optimum in E_1, then it is an optimal consumption trajectory in $\mathcal{E}$, and c_0 is an optimum in E_0.
Conversely, let c_0 be an optimum in E_0, then there exists a consumption trajectory that is optimal in $\mathcal{E}$ (and in E_1).

Example. Let a consumption trajectory c^T be P-optimal and such that it is a ray from the origin, i.e.:

$$c_t := \psi_t(c_0) = \psi^t c_0.$$

Then the investment correspondence I, for $Y_{t,0} = Y^t$ is equal to:

$$\begin{aligned} I(c^T_{-0}) &= Y^{-1}(\psi c_0 + Y^{-1}(\psi^2 c_0 + \dots + Y^{-1}(\psi^T c_0))) \\ &= \psi\, Y^{-1}(c_0 + \psi\, Y^{-1}(c_0 + \dots + \psi\, Y^{-1}(c_0))) \\ &\supset \sum_{t=1}^{T} (\psi\, Y^{-1})^t (c_0), \end{aligned}$$

The inclusion follows from Y being superadditive. It reduces to equality if Y is a linear transformation, meeting invertability and convergence conditions. (See also fig. 9.f.). The example shows how specific a proportional growth is, compared to solution based on $R^T_{-0}(c_0)$ above.

Dual economies $\mathcal{E}^*$, E^*_1 and E^*_0 can be derived which may have feasible and optimal trajectories p^T, or (p^T, π^T), depending om the structure of $P(c^T)$ and $\mathcal{C}(y_0)$. This has yet to be analysed.

Finally, if the preference relation $P(c^T)$ is defined as a cartesian product $\prod_{t=0}^{T} P_t$, then no intertemporal comparison can be made and the resulting optimum can be interpreted as an (intertemporal) Pareto optimum.

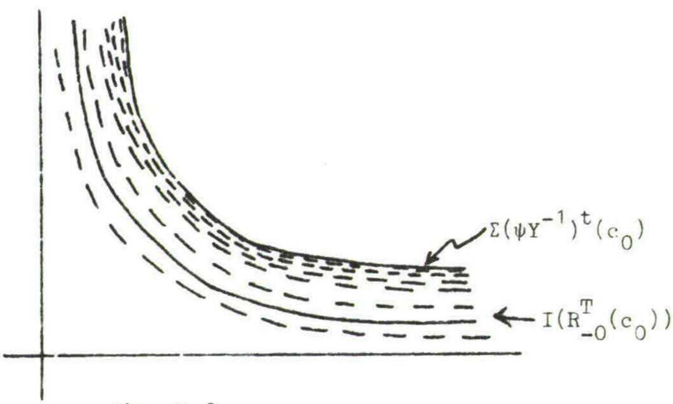

fig. 9.f.

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The dynamics of concave input/output processes.

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Representing production by a set $S \subset R_+^m \times R_+^m$ of feasible input/output combinations and associating with each pair $(x,y) \in S$ a utility value $\mu(x; y)$, we call $\mu: S \rightarrow R^1$ an input/output process if a "free disposal" assumption is satisfied. If specific property of the concept is, that, combining any number of I/O-processes in any sensible way, the logical structure is preserved. Moreover, a duality transformation is introduced possessing the same structure. Because of this structure a strong dynamic optimization theory can be developed.

1. Concave input/output processes; logic and economic relevance.

In the economic theory, a production process is considered as a process or as a complex of processes, transforming a bundle of commodities - called inputs - into an other bundle of commodities - called outputs - . It is supposed that there is only a finite number of different types of commodities; m with respect to inputs and n with respect to outputs. Thus inputs may be represented by an $x \in R_+^m$ (*) and outputs by a vector $y \in R_+^n$. With these conventions, the production process is characterized by its set of feasible input/output combinations: $S \subset R_+^m \times R_+^n$; i.e. in a pair $(x,y) \in S$, vector x is taken as the input-vector and y is taken as the output-vector. With respect to such a set $S \subset R_+^m \times R_+^n$ - in this context called a production set - we shall impose some conditions, emphasizing the nature of production.

(*) List of simbols is added at the final page.

Following the classic theory we adopt the "free disposal possibility", i.e. with respect to every feasible input/output combination $(x,y) \in S$, it is supposed that every $(\bar{x},\bar{y}) \in R_+^m \times R_+^n$ with $\bar{x} \geqq x$, $\bar{y} \leqq y$, is a feasible input/output combination, as well. Further, we assume that the production set is convex.

In a similar manner consumptive activities of an individual (or any group of individuals), is considered as a process of transforming commodities (like food, housing, etc.) into an other bundle (for instance containing labor); the set of feasible input/output combinations $C \subset R_+^m \times R_+^n$ is called the consumption set. In contrast with production, we assume here that there is a preference ordening of C, expressed by an utility function $\mu: C \to R^1$; i.e. a pair $(\bar{x},\bar{y}) \in C$ is prefered over $(\tilde{x},\tilde{y}) \in C$, if and only if $\mu(\bar{x};\bar{y}) > \mu(\tilde{x};\tilde{y})$. Thus a consumer is represented simply by a function $\mu: C \to R^1$. In this case the free disposal assumption includes that for any $(x,y) \in C$, $x \in R^m$, $y \in R^n$, a combination $\bar{x} \in R_+^m$, $\bar{y} \in R_+^n$ with $\bar{x} \geqq x$, $\bar{y} \leqq y$ is a feasible combination, as well, while in addition $\mu(\bar{x};\bar{y}) \geqq \mu(x;y)$. Further, we assume that $\mu: C \to R^1$ is a concave function.

Obvious a production process might be represented by a function $\phi: F \subset R_+^m \times R_+^n \to R^1$, as well by defining ϕ identical to zero. For that reason we shall introduce the general concept of concave input/output process, which will cover both the production aspects and the consumptive aspects in economic modelling.

1.1. The concept of concave input/output process.

Formally, we define an input/output process (abbriviated I/O-process) as a function $\mu: S \subset R^m \times R^n \to R^1$ satisfying:

(1) $S \subset R_+^m \times R_+^n$,

(2) $(x,y) \in S$, $x \in R^m$, $y \in R^n$ implies that, for every $\bar{x} \in R_+^m$, $\bar{y} \in R_+^n$ with $\bar{x} \geqq x$, $\bar{y} \leqq y$: $(\bar{x},\bar{y}) \in S$, $\mu(\bar{x};\bar{y}) \geqq \mu(x;y)$.

The function is called a concave I/O-process if the conditions (1) and (2) are satisfied and, in addition, the function is concave.

We concieve $\mu: S \subset R^m \times R^n \to R^1$ as a bi-function; that is a function where the argument is partioned into two parts, in this case $x \in R^m, y \in R^n$ together composing the argument (x,y) (cf. Rockafellar [11]). The domain with respect to argument 1 - denoted $\mathbf{D}_1(S)$ - is the set $\{x \in R^m \mid \exists y \in R^n: (x,y) \in S\}$. In a symmetric manner we have the domain in argument 2, denoted $\mathbf{D}_2(S)$. For any fixed $x \in \mathbf{D}_1(S)$, we have a "partial" function $\mu(x;\cdot)$ on the set $\{y \in R^n \mid (x,y) \in S\}$; in that we shall write $\mu(x;\cdot): S \to R^1$. Changing the arguments, one has the partial functions $\mu(\cdot;y): S \to R^1$, y being fixed in $\mathbf{D}_2(S)$.

We shall indicate a bi-function of this type by the notation $(\mu: S \to R^1,\ m{\times}n)$; in case the bi-function is improper by $(\mu: S \to [-\infty,+\infty],\ m{\times}n)$, etc.

Before illustrating the generality of the concept in economic modelling, we present an useful property concerning translation, being straightforward consequence of the definition.

1.2. Translating I/O-processes.

Let $(\mu: S \to R^1,\ m{\times}n)$ be an I/O-process, let $(a,b) \in R_+^n \times R_+^m$, and let

$(\tilde{\mu}: \tilde{S} \to R^1, m\times n)$ be a bi-function such that $\text{hypo}(\tilde{S};\tilde{\mu}) = \text{hypo}(S;\mu) +$ $+ \{a\}\times[0,b]\times\{0\}$. Then:

(1) $(\tilde{\mu}: \tilde{S} \to R^1, m\times n)$ is an I/O-process.

(2) $\forall\ (x,y) \in S$: $(x+a,y+b) \in \tilde{S}$, $\tilde{\mu}(x+a;y+b) = \mu(x;y)$

(3) $\forall\ (\tilde{x},\tilde{y}) \in S$: $(\tilde{x}-a, {}^{+}(\tilde{y}-b)) \in S$, $\tilde{\mu}(\tilde{x};\tilde{y}) = \mu(\tilde{x}-a; {}^{+}(\tilde{y}-b))$.

1.3. I/O-processes and production processes specified by a function.

In the classic theory, production is specified mostly by a "production function" $F: R_+^m \to R_+^n$ giving the outputs as an increasing function of the inputs; i.e. for every $\bar{x},\tilde{x} \in R_+^m$, with $\bar{x} \geqq \tilde{x}$, one has $F(\bar{x}) \geqq F(\tilde{x})$. Then defining

$$(1) \quad \begin{cases} S := \{(x,y) \in R_+^m \times R_+^n \mid y \leqq F(x)\}, \\ \mu: S \to R^1, \quad \mu(x;y) := 0 \quad \text{for all } (x,y) \in S, \end{cases}$$

we obtain an I/O-process. Next, concavity of F implies that $(\mu: S \to R^1, \ m\times n)$ is a concave I/O-process, indeed.

The other way around, one may specify production by putting the inputs as a increasing function $G: R_+^n \to R_+^m$ of the outputs.
Now, defining $(\mu: S \to R^1, \ m\times n)$ by

$$(2) \quad \begin{cases} S := \{(x,y) \in R_+^m \times R_+^n \mid G(y) \leqq x\}, \\ \mu(x;y) := 0. \end{cases}$$

We arrive at an I/O-process, again. If G is convex one has a concave I/O-process.

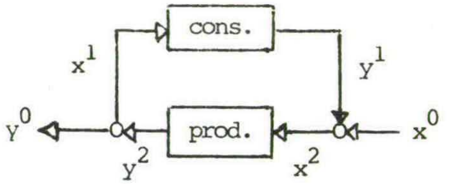
Finally, we have the standard representation of the linear activity analyses. Here the production process is supposed to be composed by a finite number (k) of sub-processes. One can operate these sub-processes at any "intensity level"; the quantities of commodities involved in the inputs and in the outputs of each sub-process is proportional with respect to its intensity level. Now, representing the set of intensity levels by R^k_+, the j-th component of an $r \in R^k_+$ being the operation level of the process numbered j, and representing the input and output rates by a nonnegative m×k-matrix A of "input-coefficients" and a non-negative n×k-matrix B of "output-coefficients", one has for any intensity $r \in R^k_+$ the corresponding inputs and outputs by Ar and Br resp. In this case $(\mu: S \to R^1,\ m{\times}n)$ with

$$(3)\qquad \begin{cases} S := \{(x,y) \in R^n_+ \times R^m_+ \mid \exists r \in R^k_+ :\ x \geqq Ar,\quad y = Br\}, \\ \mu(x;y) := 0, \end{cases}$$

evidently is the concave I/O-process, representing the production set under free disposal.

1.4. Composing concave I/O-processes.

As an illustration that the concept of concave I/O-process is extremely flexible with respect to combining, we consider a system where the consumptive activities are specified by a convex I/O-process $(\mu: C \to R^1,\ m{\times}n)$, and where the production process is given by a convex set $P \subset R^m_+ \times R^n_+$ with the free disposal facilities. Let us compose these processes according the scheme:

(1) 

$$(x^1,y^1) \in C$$
$$(x^2,y^2) \in P$$

Formally, the composed process is defined by a bi-function
$(\phi\colon S \to]-\infty,+\infty],\ \ m\times n)$:

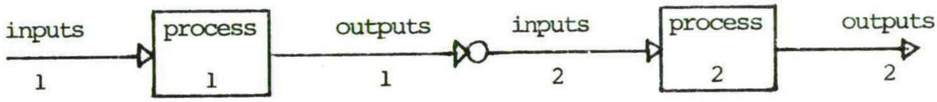
$$(2)\quad \begin{cases} S := \{(x^0,y^0) \in R^m_+ \times R^n_+ \mid \exists (x^1,y^1) \in C,\ \ (x^2,y^2) \in P: \\ \qquad\qquad\qquad\qquad x^2-y^1 \leqq x^0,\ \ y^2-y^1 \geqq y^0\} \\ \phi(x^0;y^0) := \sup \mu(x^1;y^1),\ \ \text{over } (x^1,y^1) \in C,\ \ (x^2,y^2) \in P, \\ \qquad\qquad\qquad\qquad \text{s.t. } x^2-y^1 \leqq x^0,\ \ y^2-x^1 \geqq y^0. \end{cases}$$

If, for every $(x^0,y^0) \in S$ the value of the function is finite, it is elementary to verify that the conditions 1.1.-(1), and -(2) are satisfied. Usually, under economic relevant assumptions the finiteness of the function values can be affirmated; for instance under certain boundedness conditions. Thus, again we have a concave I/O-process.

In fact, one may combine a finite number of concave I/O-processes in any manner, provided the signs of inputs and outputs are placed in the physical correct way and provided some reasonable conditions concerning boudedness are satisfies, the result will be a concave I/O-process, again. For that reason the concave I/O-process might be considered as a strongly unifying concept in micro-economic modelling under the free disposal assumption. Now we shall consider another composition, which constitutes in fact the central theme of this study.

1.5. Two stage compositions.

We consider special compositions as suggested by the scheme:

(1) 

the formal apparatus is constituted by associated with any pair of bi-functions $(f: F \to]-\infty,+\infty], m\times k)$ and $(g: G \to]-\infty,+\infty], k\times n)$ a bi-function denoted $((f \uparrow g): F\otimes G \to]-\infty,+\infty], m\times n)$:

$$(2) \begin{cases} F\otimes G: = \{(x,z) \in R^m\times R^n \mid \exists y \in R^k: (x,y) \in F, (y,z) \in G\} \\ (f\uparrow g)(x;z): = \sup f(x;y) + g(y;z), \\ \qquad \text{over } y \in R^k, \text{ s.t. } (x,y) \in F, (y,z) \in G \end{cases}$$

provided $F\otimes G \neq \emptyset$. This linking system may be extended in a inductive manner, by adding a third bi-function $(p: P \to]-\infty,+\infty], n\times j)$ resulting into the bi-function $((f\uparrow g\uparrow p): F\otimes G\otimes P \to]-\infty,+\infty], m\times j)$ defined by:
$((f\uparrow g)\uparrow p): (F\otimes G)\otimes P \to]-\infty,+\infty], m\times j)$ in case $(F\otimes G)\otimes P \neq \emptyset$.

In contrast with this "maxi-oriented" linking operator, one may introduce a "min-oriented" linking operator by associating with any pair $(p: P \to [-\infty,+\infty[, m\times k)$, $(q: Q \to [-\infty,+\infty[, k\times n)$ a bi-function $((p\downarrow q): P\otimes Q \to R^1, m\times n)$:

$$(3) \begin{cases} P\otimes Q: =\{(u,w) \in R^m\times R^n \mid \exists v \in R^k: (u,v) \in P, (v,w) \in Q\} \\ (p\downarrow g)(u,w): = \inf p(u;v) + g(v;w), \\ \qquad \text{over } v \in R^k, \text{ s.t. } (u,v) \in P, (v,w) \in Q, \end{cases}$$

provided $P\otimes Q \neq \emptyset$. Obviously:

(4) $(p{\downarrow}q)(u,w) = -\,(-p{\uparrow}-q)(u,w)$, for all $(u,w) \in P \otimes Q$.

Concerning concave I/O-processes one may verify that, the max-oriented linking operator processes the following properties.

<u>1.6. Proposition, (two stage composition of I/O-processes).</u>

Let $(f\colon F \to R^1,\ m{\times}k)$ and let $(g\colon G \to R^1,\ k{\times}n)$ be two concave I/O-processes, so that $F \otimes G \neq \emptyset$.

(1) If $(f{\uparrow}g)(F \otimes G) \subset R^1$, then $((f{\uparrow}g)\colon F \otimes G \to R^1,\ m{\times}n)$ is a concave I/O-process.

(2) One has $(f{\uparrow}g)(F \otimes G) \subset R^1$ if, and only if, for some $(\tilde{x},\tilde{z}) \in \operatorname{rint}(F \otimes G)$:

$$(f{\uparrow}g)(\tilde{x};\tilde{z}) \in R^1.$$

(3) For any $(x,z) \in F \otimes G$, the set

$$\hat{Y}_{x,z} := \{\hat{y} \in R^k \mid (x,\hat{y}) \in F,\ (\hat{y},z) \in G,\ f(x;\hat{y}) + g(\hat{y};z) \geqq (f{\uparrow}g)(x;z)\}$$

is convex (possible empty).

2. Valuation, prices, and the dual input/output process.

Let us denote the input/output prices concerning an I/O-process (μ: $S \to R^1$, m×n) by $p \in R_+^m$, $q \in r_+^n$ resp. Then the net value of an input/output combination $(x,y) \in S$ is $\mu(x;y) - \langle p,x\rangle + \langle q,y\rangle$. Thus value maximization gives rise to the following max-oriented transformations.

2.1. Duals. With any bi-function (μ: $S \to R^1$, m×n) we associate:

(1) The max-oriented) dual ($\underline{\Delta}\mu$: $\underline{\Delta}S \to R^1$, m×n) by:

$\underline{\Delta}\mu(u;v) := \sup \mu(x;y) - \langle u,x\rangle + \langle v,y\rangle$, over $(x,y) \in S$,

$\underline{\Delta}S := \{(u,v) \in R_+^m \times R_+^n | \underline{\Delta}\mu(u;v) < +\infty\}$. (provided $\underline{\Delta}S \neq \emptyset$)

(2) The (partial) (max-oriented) dual in argument 1: ($\overset{\Delta}{_1}\mu$: $\overset{\Delta}{_1}S \to R^1$, m×n),

$\overset{\Delta}{_1}\mu(u;y) := \sup \mu(x;y) - \langle u,x\rangle$, over $x \in R^m$, s.t. $(x,y) \in S$

$\overset{\Delta}{_1}S := \{(u;y) \in R_+^m \times D_2(S) | \overset{\Delta}{_1}\mu(u;y) < +\infty\}$ (provided $\overset{\Delta}{_1}S \neq \emptyset$).

(3) The (partial) (min-oriented) dual in argument 2: ($\overset{\Delta}{_2}\mu$: $\overset{\Delta}{_2}S \to R^1$, m n),

$\overset{\Delta}{_2}\mu(x;v) := \sup \mu(x;y) + \langle v,y\rangle$, over $y \in R^n$, s.t. $(x,y) \in S$.

$\overset{\Delta}{_2}S := \{(x,v) \in D_1(S) \times R_+^n | \overset{\Delta}{_2}\mu(x;v) < +\infty\}$ (provided $\overset{\Delta}{_2}S \neq \emptyset$).

In the context of I/O-processes the meaning of these concepts should be clear: (1) represents maximization of the net value of the process, given input prices $u \in R_+^m$ and output prices $v \in R_+^n$; and $\underline{\Delta}S$ is the set of input/output prices so that the net value of the process is bounded, etc.

Provided $\underline{\Delta}S \neq \emptyset$, the dual of I/O-processes have a particular property which is an immediate consequence of the free disposal assumption:

2.2. Proposition (price monotonicity property).

Let (μ: $S \to R^1$, m×n) be an I/O-process. Suppose $\underline{\Delta}S \neq \emptyset$, and let

$(p,q) \in \underline{\Delta}S$. Then for every $\bar{p} \in R^m_+$, $\bar{p} \geqq p$, $\bar{q} \in R^n_+$, $\bar{q} \leqq q$, one has:

(1) $(\bar{p},\bar{q}) \in \underline{\Delta}S$, $\underline{\Delta}\mu(\bar{p};\bar{q}) \leqq \underline{\Delta}(p;q)$.

(2) $\forall\, y \in D_2(S)$: ${}^{\Delta}_{1}\mu(\bar{p};y) \leqq {}^{\Delta}_{1}\mu(p;y)$.

(3) $\forall\, x \in D_1(S)$: ${}^{\Delta}_{2}\mu(x;\bar{q}) \leqq {}^{\Delta}_{2}\mu(x;q)$.

Thus increasing the input prices and/or decreasing the output prices, the value of the I/O-process will not increase.

2.3. Dual functions and Rockafellar's adjoint of a bi-function.

The formal relations between a concave I/O-process and its duals become clear by recognizing that, as a matter of fact, the dual is a modified Frenchel-transformation, closely related to Rochafellars adjoint of a bi-function (cf. [11]. §30). There - in a different notation - the max-oriented adjoint of a bi-function ($\mu: S \to R^1$, m×n) is a bi-function ($\underline{\wedge}\mu: \underline{\wedge}S \to R^1$, m×n) defined by:

$$(1)\quad \begin{cases} \underline{\wedge}\mu(u;v) := \sup\ \mu(x;y) - \langle u,x\rangle + \langle v,y\rangle, \\ \qquad\qquad \text{over } (x,y) \in S,\ x \in R^m,\ y \in R^n, \\ \underline{\wedge}S := \{(u,v) \in R^m \times R^n \mid \underline{\wedge}\mu(u;v) < +\infty\}. \end{cases}$$

So, the only difference with respect to our dual function is, that we restrict the domain to $R^m_+ \times R^n_+$, which is justified by the price interpretation of the concept. However, especially for the I/O-process there is a simple one-to-one relation between Rockafellar's adjoint and our dual. Namely: denoting with respect to any $z \in R^k$ vectors ${}^{+}(z), {}^{-}(z) \subset R^k_+$ as the non-negative pair so that $z = {}^{+}(z) - {}^{-}(z)$, a straightforward consequences of 1.1 -(1) and -(2) are:

(2) $\forall\ (u,v) \in \underline{\wedge}S,\ u \in R^m,\ v \in R^n$: $(u,{}^+(v)) \in \underline{\Delta}S$, $\underline{\wedge}\mu(u;v) = \underline{\Delta}\mu(u;{}^+(v))$.

(3) $\forall\ (p,q) \in \underline{\Delta}S,\ p \in R^m,\ q \in R^n$: $(p,q) \in \underline{\wedge}S$, $\underline{\wedge}\mu(p;q) = \underline{\Delta}\mu(p;q)$.

On the other hand, if for any $(\mu\colon S \to R^1,\ m\times n)$ with the property that given $(x,y) \in S$, $x \in R^m$, $y \in R^n$, for every $\bar{x} \geqq x$, $\bar{y} \leqq y$: $(\bar{x},\bar{y}) \in S$, $\mu(\bar{x};{}^+(\bar{y})) = \mu(\bar{x};\bar{y}) \geqq \mu(x;y)$, then:

(4) $\underline{\wedge}S = \underline{\Delta}S$, $\forall\ (u,v) \in \underline{\wedge}S$: $\underline{\wedge}\mu(u;v) = \underline{\Delta}\mu(u;v)$.

These relations imply that the theory on Rockafellar's adjoint is fully applicable on our dual functions, provided we restrict ourselves to I/O-processes.

As a first observation: the adjoint of any bi-function is convex bi-function (provided the domain is not empty), further, concavity of the bi-function implies that $\underline{\wedge}S \neq \emptyset$. Since the similar holds for the dual of an I/O-process, and since we have the monotonity property 2.2, we may conclude that for every concave I/O-process $(\mu\colon S \to R^1,\ m\times n)$ the negative dual $(-\underline{\Delta}\mu\colon \underline{\Delta}S \to R^1,\ m\times n)$ is a concave I/O-process. This justifies the introduction of the following concept.

2.4. The dual I/O-process.

We define a dual I/O-process as a bi-function $(\nu\colon W \to R^1,\ m\times n)$ satisfying the conditions:

(1) $W \subset R^m_+ \times R^n_+$.

(2) $(u,v) \in W$; $u \in R^m$, $v \in R^n$ implies for every $\bar{u} \in R^m_+$, $\bar{u} \geqq u$, $\bar{v} \in R^n_+$, $\bar{v} \leqq v$: $(\bar{u},\bar{v}) \in W$, $\nu(\bar{u};\bar{v}) \leqq \nu(u;v)$.

(3) $\nu\colon W \to R^1$ is a convex function.

Thus, the dual of an I/O-process is a dual I/O-process, indeed. Observe that a bi-function is a dual I/O-process only if its negative is a concave I/O-process. Corresponding to this opposite orientation, we introduce the following minimum oriented duals:

2.5. Min-oriented duals.

With respect to any $(\nu\colon W \to R^1,\ m\times n)$ we associate:

(1) The min-oriented dual: $(\underline{\nabla}\nu\colon \underline{\nabla}W \to R^1,\ m\times n)$,

$$\underline{\nabla}\nu(x;y) := \inf\ \nu(u;v) + \langle x,u\rangle - \langle y,v\rangle,$$
$$\text{over } (u,v) \in W,\ u \in R^m,\ v \in R^n,$$
$$\underline{\nabla}W := \{(x,y) \in R^m_+ \times R^n_+ \mid \underline{\nabla}\nu(x;y) > -\infty\} \text{ (provided } \underline{\nabla}W \neq \emptyset).$$

(2) The (partial) min-oriented dual in arg. 1: $(\overset{\nabla}{_1}\nu\colon \overset{\nabla}{_1}W \to R^1,\ m\times n)$

$$\overset{\nabla}{_1}\nu(x;v) := \inf\ \nu(u;v) + \langle x,u\rangle,$$
$$\text{over } u \in R^m,\ \text{s.t. } (u,v) \in W,$$
$$\overset{\nabla}{_1}W := \{(x,v) \in R^m \times D_2(W) \mid \overset{\nabla}{_1}\nu(x;v) > -\infty\}.$$

(3) The (partial) min-oriented dual in arg. 2: $(\overset{\nabla}{_2}\nu\colon \overset{\nabla}{_2}W \to R^1,\ m\times n)$, defined in the symmetric manner.

The corresponding min-oriented adjoint (cf. [11], §30) - we use the notation $(\underline{\vee}\nu\colon \underline{\vee}W \to R^1,\ m\times n)$ - is, omitting the non-negativity condition in $\underline{\nabla}W$, defined as the min-oriented dual. For an I/O-process the relations between the max-oriented dual and the min-oriented duals are the same as those between Rockafellar's max-oriented adjoints and his min-oriented adjoint. We will formulate some of these relations with the help of a particular continuity concept; namely: a function $f\colon S \subset R^n \to R^1$ is called lower-continuous if its hypograph (i.e. $\mathrm{hypo}(S;f) := \{(x,\nu) \in S\times R^1 \mid \nu \leqq f(x)\}$) is closed; the function is called <u>lower-continuous at a point</u> $x \in S$ if $\{\alpha \in R^1 | \ (x,\alpha) \in \mathrm{cl}(\mathrm{hypo}(S;f))\} =]-\infty, f(x)]$. In a similar manner <u>upper-continuity</u> is defined with respect to the epigraph of the function; that is the set $\mathrm{epi}(S;f) := \{(x,\nu) \in S \times R^1 | \ \nu \geqq f(x)\}$. Note that a function $f: S \subset R^n \to R^1$ is continue at certain $x \in S$, if and only if he is both lower- and upper-continuous at x. Now, starting with some properties concerning the domain of the duals, we shall list some well-known properties (cf. [11]) of Rockafellar's adjoints in terms of our duals.

<u>2.6. Proposition.</u> Let $(\mu: S \to R^1, m\ n)$ be a concave I/O-process. Then:

(1) $\underline{\Delta}S \neq \emptyset$, $(\underline{\Delta}\mu: \underline{\Delta}S \to R^1, m{\times}n)$ is an upper-continuous dual I/O-process.

(2) $\mathbb{D}_1(\underline{\Delta}S) \times \mathbb{D}_2(S) \subset \overset{\Delta}{_1}S \subset \mathrm{cl}(\mathbb{D}_1(\underline{\Delta}S)) \times \mathbb{D}_2(S)$.

(3) $\forall\ y \in \mathrm{rint}(\mathbb{D}_2(S))$: $\{u \in R^m | (u,y) \in \overset{\Delta}{_1}S\} = \mathbb{D}_1(\underline{\Delta}S)$.

(4) $\forall\ (\bar{u},\bar{v}) \in \underline{\Delta}S$, $\bar{u} \in R^m$, $\bar{v} \in R^n$:

$$\underline{\Delta}\mu(\bar{u};\bar{v}) = \sup \overset{\Delta}{_1}\mu(\bar{u};y) + \langle\bar{v},y\rangle, \text{ over } y \in \mathbb{D}_2(S).$$

In the symmetric manner -(2), (3), and -(4) also hold with respect to the dual in arg. 2. In fact, this can be said of all statements concerning partial functions and partial duals.

<u>2.7. Proposition.</u> Let $(\mu: S \to R^1, m{\times}n)$ be a concave I/O-process. Suppose the function is lower-continuous at $(\bar{x},\bar{y}) \in S$, $\bar{x} \in R^m$, $\bar{y} \in R^n$. Then:

(1) $\mu(\bar{x};\bar{y}) = \inf \underline{\Delta}\mu(u;v) + \langle\bar{x},u\rangle - \langle\bar{y},v\rangle$,

$$\text{over } (u,v) \in S,\ u \in R^m,\ v \in R^n.$$

(2) $\mu(\bar{x};\bar{y}) = \inf \overset{\Delta}{_1}\mu(u;\bar{y}) + \langle\bar{x},u\rangle$, over $u \in \mathbb{D}_1(\underline{\Delta}S)$.

(3) $\forall\ \bar{u} \in \mathbb{D}_1(\underline{\Delta}S)$: $\overset{\Delta}{_1}\mu(\bar{u};\bar{y}) = \inf \underline{\Delta}\mu(\bar{u};v) - \langle\bar{y},v\rangle$,

$$\text{over } v \in R^n, \text{ s.t. } (\bar{u},v) \in \underline{\Delta}S.$$

Obviously a consequence of 2.7 -(1) is that the min-oriented dual of the max-oriented dual of a lower-continuous concave I/O-process, is equivalent to that I/O-process. In fact, this property exclusively holds for lower-continuous concave I/O-processes.

2.8. Proposition. Let (μ: $S \to R^1$, m×n) be a concave I/O-process. Then for every $(\tilde{x},\tilde{y}) \in \mathrm{rint}(S)$ there is an $(\bar{u},\bar{v}) \in \underline{\Delta}S$ so that $\underline{\Delta}\mu(\bar{u};\bar{v}) = \mu(\tilde{x};\tilde{y}) - \langle\bar{u},\tilde{x}\rangle + \langle\bar{v},\tilde{y}\rangle$.

2.9. Proposition. Let (μ: $S \to R^1$, m×n) be a lower continuous concave I/O-process. Then:

(1) $\forall\ (\tilde{u},\tilde{v}) \in \mathrm{rint}(\underline{\Delta}S): \exists (\bar{x},\bar{y}) \in S$:

$$\mu(\bar{x};\bar{y}) = \underline{\Delta}\mu(\tilde{u};\tilde{v}) + \langle\tilde{u},\bar{x}\rangle - \langle\tilde{v},\bar{y}\rangle.$$

(2) $\forall\ y \in \mathrm{rint}(\mathbf{D}_2(S)),\ \tilde{u} \in \mathrm{rint}(\mathbf{D}_1(\underline{\Delta}S)): \exists\ \bar{x} \in R^m,\ \bar{v} \in R^n$:

$$\mu(\bar{x},\tilde{y}) \in S, \quad (\tilde{u},\bar{v}) \in \underline{\Delta}S,$$

$$\mu(\bar{x};\tilde{y}) - \langle\tilde{u},\bar{x}\rangle = \overset{\Delta}{_1}\mu(\tilde{u};\tilde{y}) = \underline{\Delta}\mu(\tilde{u};\bar{v}) - \langle\tilde{y},\bar{v}\rangle.$$

(3) $\forall\ \tilde{x} \in \mathrm{rint}\ (\mathbf{D}_1(S)),\ \tilde{v} \in \mathrm{rint}(\mathbf{D}_2(\underline{\Delta}S)): \exists\ \bar{y} \in R^n,\ \bar{u} \in R^m$:

$$(\tilde{x},\bar{y}) \in S, \quad (\bar{u},\tilde{v}) \in \underline{\Delta}S,$$

$$\mu(\tilde{x};\bar{y}) + \langle\tilde{v},\bar{y}\rangle = \overset{\Delta}{_2}\mu(\tilde{x};\tilde{v}) = \underline{\Delta}\mu(\bar{u};\tilde{v}) + \langle\bar{u},\tilde{x}\rangle.$$

The effect of scaling on the dual I/O-processes is given by the following property:

2.10. Proposition. Let (μ: $S \to R^1$, m×n) be a (lower continuous) (concave) I/O-process, with $\underline{\Delta}S \neq \emptyset$. Let $\gamma \in R^1_{++}$, and let ($\tilde{\mu}$: $\tilde{S} \to R^1$,m×n) be defined by $\tilde{\mu}(x;y) = \gamma\mu(x;y)$, $\tilde{S} := S$. Then, with respect to the dual $\underline{\Delta}\tilde{\mu}$: $\underline{\Delta}\tilde{S} \to R^1$ one has: $\underline{\Delta}\tilde{S} = \gamma\underline{\Delta}S$, $\underline{\Delta}\tilde{\mu}(u;v) = \gamma\underline{\Delta}\mu(\gamma^{-1}u;\gamma^{-1}v)$.

3. Decomposition by duality methods.

We restrict ourselves to I/O-processes being composed as described in 1.5. However similar results can be deduced for other compositions.

3.1. The dual of two-stage I/O-processes.

Let $(f: F \to R^1, m\times n)$, $(g: G \to R^1, k\times n)$ be a pair of bi-functions, and let us consider $((f\uparrow g): F\otimes G \to]-\infty,+\infty], m\times n)$ as defined in 1.5 by:

$$(1)\begin{cases} F\otimes G := \{(x,z) \in R^m\times R^n \mid y \in R^k: (x,y) \in F,\ (y,z) \in G\}, \\ (f\uparrow g)(x;z) := \sup f(x;y) + g(y;z), \\ \qquad\qquad \text{over } y \in R^k, \text{ st. } (x,y) \in F,\ (y,z) \in G \end{cases}$$

provided $F\otimes G \neq \emptyset$. Now, consider the bi-function $(\phi: S \to]-\infty,+\infty]$, $(m+k)\times(k+n))$, defined by:

$$(2)\begin{cases} S := \{(x,a,b,z) \in R^m\times R^k_+\times R^k_+\times R^n \mid y^1,y^2 \in R^k: \\ \qquad\qquad (x,y^1) \in F,\ (y^2,z) \in G,\ y^2-y^1 = a-b\}, \\ \phi(x,a;b,z) := \sup f(x;y^1)+g(y^2;z), \text{ over } y^1,y^2 \in R^k, \\ \qquad\qquad \text{st. } (x,y^1) \in F,\ (y^2,z) \in G,\ y^2-y^1 = a-b. \end{cases}$$

Clearly, for every $(x,z) \in F\otimes G$, $x \in R^m$, $y \in R^n$, $c \in R^k_+$, one has: $(x,c,c,z) \in S$, $\phi(x,c;c,z) = (f\uparrow g)(x;z)$. In addition, if f and g are (concave) I/O-processes, and if $\phi(S) \subset R^1$, one may verify that ϕ is a (concave) I/O-process. Hence, if f and g are concave I/O-processes, then one has $\phi(S) \subset R^1$ if and only if the dual of ϕ is well-defined, i.e. if $\underline{\Delta}S \neq \emptyset$. Defining the dual of ϕ directly in the constituting I/O-processes:

$$(3)\begin{cases}\underline{\Delta}\phi(x^*,a^*;b^*,z^*) := \sup\ (f(x;y^1) + g(y^2;z) - \langle x^*,x\rangle + \langle b^*,y^1\rangle - \\ \qquad\qquad - \langle a^*,y^2\rangle + \langle z^*,z\rangle + \langle b^*,b-y\rangle - \langle a^*,a-y^2\rangle), \\ \text{over } (x,y^1) \in F,\ (y^2,z) \in G,\ \ a,b \in R^k_+,\ \text{st. } b-y^1 = a-y^2, \\ \underline{\Delta}S := \{(x^*,a^*,b^*,z^*) \in R^m\times R^k\times R^k\times R^n \,|\, \underline{\Delta}\phi(x^*,a^*;b^*,z^*) < +\infty\},\end{cases}$$

it is easy to verify that:

$$(4)\begin{cases}\Delta S = \{(x^*,a^*;b^*,z^*) \in R^m\times R^k\times R^k\times R^n \,| \\ \qquad (x^*,b^*) \in \underline{\Delta}F,\quad (a^*,z^*) \in \underline{\Delta}G,\quad b^* \leqq a^*\}, \\ \underline{\Delta}\phi(x^*,a^*;b^*,z^*) = \underline{\Delta}f(x^*;b^*) + \underline{\Delta}g(a^*;z^*), \text{ with} \\ \quad (x^*,a^*,b^*,z^*) \in \underline{\Delta}S,\ x^* \in R^m,\ a^*,b^* \in R^k,\ z^* \in R^n.\end{cases}$$

Further, the definition of the max-oriented dual 2.1 -(1) with respect to ϕ, and the definition of ϕ itself, implies:

$$(5)\begin{cases}\forall\ (x,z) \in F\otimes G,\ c \in R^k_+,\ (u^*,b^*) \in \underline{\Delta}F,\ (a^*,w) \in \underline{\Delta}G \text{ with } b^* \leqq a^*: \\ (f+g)(x;z) - \langle u,x\rangle + \langle w,z\rangle + \langle b - a\ ,c\rangle \leqq \underline{\Delta}f(u;b^*) + \underline{\Delta}g(a^*;w).\end{cases}$$

In addition, by virtue of 2.8 -(1) one may conclude that, in case there is a pair $(x,a) \in F$, $(b,z) \in G$, with $a > b$:

$$(6)\begin{cases}\forall\ (x,z) \in \text{rint}(F\otimes G),\ c \in R^k_+: \exists (u,b^*) \in \underline{\Delta}F,\ (a^*,w) \in \underline{\Delta}G,\ b^* \leqq a^*: \\ (f+g)(x;z) - \langle u,x\rangle + \langle w,z\rangle + \langle b^*-a^*,c\rangle = \underline{\Delta}f(u;b^*) + \underline{\Delta}g(a^*;w).\end{cases}$$

Combining (5) and (6) it is obvious that there has to be a pair $(\hat{u},\hat{b}^*) \in \underline{\Delta}F$, $(\hat{a}^*,\hat{w}) \in \underline{\Delta}G$ with $\hat{a}^* = \hat{b}^*$, satisfying the equality in (6). Hence, defining $((\underline{\Delta}f+\underline{\Delta}g): \underline{\Delta}F\otimes\underline{\Delta}G \to [-\infty,+\infty[,\ m\times n)$ as indicated in 1.5 -(3)

by $((\underline{\Delta}f)\dagger(\underline{\Delta}g)\colon (\underline{\Delta}F)\otimes(\underline{\Delta}G) \to [-\infty,+\infty[, m\times n)$, we may conclude that $(\underline{\Delta}f\dagger\underline{\Delta}g)$ may be considered as the max-oriented dual of $(f\dagger g)$. More precisely, we have:

<u>3.2. Proposition.</u> Let $(f\colon F \to R^1, m\times k)$ and $(g\colon G \to R^1, k\times n)$ be concave I/O-processes. Then:

(1) For any $(x,y,z),(x,y) \in \mathrm{rint}(F)$, $(y,z) \in \mathrm{rint}(G)$ one has

$(f\dagger g)(x;z) \subset R^1$ if and only if $\underline{\Delta}F\otimes\underline{\Delta}G \neq \emptyset$.

(2) $\forall\ (x,z) \in F\otimes G$, $(u,w) \in \underline{\Delta}F\otimes\underline{\Delta}G$:

$(f\dagger g)(x;z) - \langle u,x\rangle + \langle w,z\rangle \leqq (\underline{\Delta}f\dagger\underline{\Delta}g)(u,w)$.

(3) $\forall\ (x,y,z) \mid (x,y) \in \mathrm{rint}(F), (y,z) \in \mathrm{rint}(G)\colon \exists\, (u,v,w) \in R^m\times R^k\times R^n$

$(u,v) \in \underline{\Delta}F$, $(v,w) \in \underline{\Delta}G$,

$(f\dagger g)(x;z) - \langle u,x\rangle + \langle w,z\rangle = \underline{\Delta}f(u;v) + \underline{\Delta}g(v;w)$.

<u>3.3. Symmetry of the duality relations.</u>

A remarkable aspect of these results is that the duality concepts of $(f\dagger g)$ lead to a bi-function defined by:

$$(1)\quad \begin{cases} \underline{\Delta}F\otimes\underline{\Delta}G := \{(u,w) \in R^m\times R^n \mid \exists\, v \in R^k\colon (u,v) \in \underline{\Delta}F,\ (v,w) \in \underline{\Delta}G\}, \\ (\underline{\Delta}f\dagger\underline{\Delta}g)(u;w) := \inf \underline{\Delta}f(u;v) + \underline{\Delta}g(v;w), \\ \qquad\qquad\qquad \text{over } v \in R^k, \text{ st. } (u,v) \in \underline{\Delta}F,\ (v,w) \in \underline{\Delta}G, \end{cases}$$

with properties completely symmetric with respect to our starting point 3.1 -(1). I.e. starting from 3.3 -(1) and following the same procedure, but in the opposite orientation, one will arrive at the bi-function $((\underline{\nabla}\underline{\Delta}f\dagger\underline{\nabla}\underline{\Delta}g)\colon \underline{\nabla}\underline{\Delta}F\otimes\underline{\nabla}\underline{\Delta}G \to]-\infty,+\infty], m\times n)$. Hence, if f and g are lower continuous concave I/O-processes, implying that the min-oriented dual of the max-oriented dual is equivalent to the original, this procedure ends up with the original problem 3.1 -(1), again. Clearly, changing the orientation, all properties of 3.1 -(1) are fully applicable on the dual form 3.3 -(1), provided the constituting bi-functions are lower continuous concave I/O-processes. Thus the symmetric formulation of 3.2 runs as follows.

<u>3.4. Proposistion.</u> Let $(f\colon F \to R^1, m\times k)$, and $(g\colon G \to R^1, k\times n)$ be lower continuous concave I/O-processes. Then

(1) For any (u,v,w), $(u,v) \in \mathrm{rint}(\underline{\Delta}F)$, $(v,w) \in \mathrm{rint}(\Delta G)$, one has

$(\underline{\Delta}f \downarrow \underline{\Delta}g)(u;w) \subset R^1$ if and only if $F\otimes G \neq \emptyset$.

(2) $\forall\ (u,w) \in \underline{\Delta}F\otimes\underline{\Delta}G,\ (x,z) \in F\otimes G$:

$(\underline{\Delta}f \downarrow \underline{\Delta}g)(u,w) + \langle u,x\rangle - \langle w,z\rangle \geqq (f\uparrow g)(x;z)$.

(3) $\forall\ (u,v,w) \mid (u,v) \in \mathrm{rint}(\underline{\Delta}F),\ (v,w) \in \mathrm{rint}(\Delta G)\colon \exists\ (x,y,z) \in R^m\times R^k\times R^n$:

$(x,y) \in F,\ (y,z) \in G$,

$(\underline{\Delta}f \downarrow \underline{\Delta}g)(u,w) + \langle u,x\rangle - \langle w,z\rangle = f(x;y) + g(y;z)$.

Other properties can be deduced with the help of 2.9 -(2), -(3) and the properties given in 3.1 to 3.4. For instance:

<u>3.5. Proposition.</u> Let $(f\colon F \to R^1, m\times k)$ and $(g\colon G \to R^1, k\times n)$ be lower continuous concave I/O-processes. Suppose there is a pair $(x,a) \in F$, $(b,z) \in G$ with $a > b$, and pair $(u,b^*) \in \underline{\Delta}F$, $(a^*,w) \in \underline{\Delta}G$ so that $b^* > a^*$. Then:

(1) $\tilde{\forall}\ z \in \mathrm{rint}(\mathbb{D}_2(F\otimes G))$, $\tilde{u} \in \mathrm{rint}(\mathbb{D}_1^{\Delta}(\underline{\Delta}F\otimes\underline{\Delta}G))$:

$\exists(\bar{x},\bar{y}) \in F,\ (\bar{v},\bar{w}) \in \underline{\Delta}G\colon (\bar{y},\tilde{z}) \in G,\ (\tilde{u},\bar{v}) \in \underline{\Delta}F$,

$$f(\bar{x};\bar{y}) + g(\bar{y};\tilde{z}) - \langle\tilde{u},\bar{x}\rangle = {}_1^{\Delta}(f\uparrow g)(\tilde{u};\tilde{z}) =$$

$$= \underline{\Delta}f(\tilde{u};\bar{v}) + \underline{\Delta}g(\bar{v};\bar{w}) - \langle\bar{w},\tilde{z}\rangle.$$

(2) $\forall\, \tilde{x} \in \operatorname{rint}(\mathbb{D}_1(F \otimes G))$, $\tilde{w} \in \operatorname{rint}(\mathbb{D}_2(\underline{\Delta}F \otimes \underline{\Delta}G))$:

$$\exists (\bar{y},\bar{z}) \in G,\ (\bar{u},\bar{v}) \in \underline{\Delta}F:\ \ (\tilde{x},\tilde{y}) \in F,\ (\bar{v},\tilde{w}) \in \underline{\Delta}G,$$

$$f(\tilde{x};\bar{y}) + g(\bar{y};\bar{z}) + \langle\tilde{w},\bar{z}\rangle = \overset{\Delta}{}_2(f+g)(\tilde{x};\tilde{w}) =$$

$$= \underline{\Delta}f(\bar{u};\bar{v}) + \underline{\Delta}g(\bar{v};\tilde{w}) + \langle\bar{u};\tilde{x}\rangle.$$

3.6. Decomposition principle.

In an implicit form the decomposition principle is given by property 3.2. Namely, writing the equality in 3.2 -(3):

$$(1)\ \begin{cases} (\sup f(x;y) + g(y;z) - \langle u,x\rangle + \langle w,z\rangle, \\ \qquad \text{over } x,y,z, \text{ st. } (x,y) \in F,\ (y,z) \in G) = \\ = (\sup f(\tilde{x};\tilde{y}) - \langle u,\tilde{x}\rangle + \langle v,\tilde{y}\rangle, \text{ over } (\tilde{x},\tilde{y}) \in F) + \\ \ + (\sup g(\bar{y};\bar{z}) - \langle v,\bar{y}\rangle + \langle w,\bar{z}\rangle, \text{ over } (\bar{y},\bar{z}) \in G), \end{cases}$$

it appears that, given a triple (u,v,w) as mentioned in 3.2 -(3), any optimal triple $(\hat{x},\hat{y},\hat{z})$ in the lefthand-side max-problem,generates optimal solutions $(\hat{x},\hat{y})$ and $(\hat{y},\hat{z})$ with respect to resp. the first and the second max-problem in the righthand-side. If f and g are strictly concave, implying unicity of optimal solutions, then for any optimal pair $(\tilde{x},\tilde{y})$, $(\bar{y},\bar{z})$ with respect to the righthand-side, one has the equality $\tilde{y} = \bar{y}$, and the fact that $(\tilde{x},\tilde{y},\bar{z})$ constitutes an optimal solution in the lefthand-side max-problem. Thus instead of solving the two stage max-problem one may solve the separate problems. In an economic context the triple (u,v,w) with these properties might be conceived as a price-system such that maximization of the net-value of the separate processes induces a combination of inputs and outputs that fits into centralized maximization of the total system.

4. Dynamic input/output processes.

Now we shall explore the concepts of I/O-process and two-stage linking in order to study an economic system where the activities take place during a sequence - finite or infinite - of periods, in such a manner that inputs at the beginning of a period result into outputs which become available at the end of that period. These outputs are used for the activities at the succeeding period, and so on.

The periods are numbered $t = 0,1,\dots,h$. Period 0 is considered as the last passed period. Period h is the final period under consideration; it is called the (time)-horizon. If the horizon is not specified, we shall speak of an infinite (or open) horizon system. The moments of period changing are called time-points; to be indicated as "the start of period t" or as "the end of period t".

Specifying the activities and the corresponding utilities by a sequence of I/O-processes, the assumptions lead to the following formal structure.

4.1. Definition of dynamic I/O-processes.

Let $(\mu^t: S^t \to R^1,\ m_t \times n_t)$, $t = 1,\dots,h$ be a sequence of bi-function (h being positive integer or positive infinite). We shall call this sequence a dynamic (lower continuous) (concave) I/O-process, if:

(1) each $\mu^t: S^t \to R^1$ is a (lower continuous) (concave) I/O-process.

(2) $m_{t+1} = n_t$, $t = 1,\dots,h-1$.

(3) there is an $\bar{m}$ such that $m_t \leqq \bar{m}$, $t = 1,\dots,h$; clearly, this condition is relevant in case $h := +\infty$.

A sequence $\{(x^t,y^t)\}_1^h$ is called a feasible path if $(x^t,y^t) \in S^t$,

$t = 1,\dots,h$, $y^t = x^{t-1}$, $t = 2,\dots,h$. Consequently, a dynamic I/O-process is called feasible if there is feasible path; a vector $\tilde{x}$ is called a <u>feasible initial state</u> (or <u>input</u>) if there is a feasible path $\{(x^t,y^t)\}_1^h$ with $x^1 := \tilde{x}$. We start with discussing some properties concerning translation, duality and optimality aspects. Again it will appear that the concept is very flexible, the more because each of the separate I/O-processes might be composed of a number of I/O-processes.

4.2. Translation dynamic I/O-processes.

Let $(\mu^t\colon S^t \to R^1, m_t \times n_t)$ be a dynamic I/O-process. Let $\{r^t\}_1^{h+1}$ a sequence of vectors, $r^t \in R^{m_k}$, $t = 1,\dots,h$, $r^{h+1} \in R_+^{n_h}$. Then defining $(\tilde{\mu}^t\colon \tilde{S}^t \to R^1, m_t \times n_t)$, $t = 1,\dots,h$, so that for each t:

$\text{hypo}(\tilde{S}^t; \tilde{\mu}^t) = \text{hypo}(S^t; \mu^t) + \{\,{}^-(r^t)\}\times[0, {}^+(r^{t+1})]\times\{0\}$ one has:

(1) $(\tilde{\mu}^t\colon \tilde{S}^t \to R^1, m_t \times n_t)$, $t = 1,\dots,h$ is a dynamic I/O-process.

(2) $\{(x^t,y^t)\}_1^h$ satisfies $(x^t,y^t) \in S^t$, $t = 1,\dots,h$,
$x^t - y^{t-1} = r^t$, $r = 2,\dots,h$, if and only if
$\{(x^t + {}^-(r^t),\ y^t + {}^+(r^{t+1}))\}_1^h$ is a feasible path with respect to the dynamic I/O-process in (1); in that case:
$\mu^t(x^t;y^t) = \tilde{\mu}^t(x^t + {}^-(r^t);\ y^t + {}^+(r^{t+1}))$, $t = 1,\dots,h$.

The consequence of this property is that, studying the dynamics, we may restrict ourselves to feasible paths in the sense of 4.1, without loss of generality.

4.3. Optimality aspects.

Generally, there is no natural manner to compair the utility values of the separate periods. A usefull, but weak, optimality criterium is given by the concept of Pareto-efficiency; in this context we shall call a feasible path $\{(\hat{x}^t,\hat{y}^t)\}_1^h$, given $\hat{x}^1$ and $\hat{y}^h$, Pareto-efficient, if no feasible path $\{(x^t,y^t)\}_1^h$, $x^1 := \hat{x}^1$, $y^h := \hat{y}^h$ exists so that $\mu^t(x^t;y^t) \geqq \mu^t(\hat{x}^t;\hat{y}^t)$, $t = 1,2,\dots,h$, with at least for one period the strict inequality.

Here we shall restrict ourselves to the "weighted sum" objective function $\Sigma_{t=1}^h \gamma_t \mu^t(x^t;y^t)$, where (tentative) $h < +\infty$ and where $\{\gamma_t\}_1^h$ are positive scalars. Thus we arrive at, what we shall call, the fixed end point and the free end point dynamic max-problems:

(1) $\bar{\phi}^h(x^1;y^h) := \sup \Sigma_{t=1}^h \gamma_t \mu^t(x^t;y^t)$,
over $\{x^t\}_2^h$, $\{y^t\}_1^{h-1}$, st. (given x^1, y^h),
$(x^t,y^t) \in S^t$, $t = 1,\dots,h$, $x^{t+1} = y^t$, $t = 1,\dots,h-1$.

(2) $\tilde{\phi}^h(x^1;v^h) := \sup \gamma_h \langle v^h,y^h \rangle + \Sigma_{t=1}^h \gamma_t \mu(x^t;y^t)$,
over $\{x^t\}_2^h$, $\{y^t\}_1^h$, st. (given x^1, v^h),
$(x^t,y^t) \in S^t$, $t = 1,\dots,h$, $x^{t+1} = y^t$, $t = 1,\dots,h-1$.

One may verify that, appart from the terminal outputs y^h, any optimal path in (1) or (2) is Pareto-efficient, as well. A rigorous analysis on this matter is given by Koopmans [9].

4.4. Dual dynamic I/O-processes.

Denoting the set of admissible pairs of initial inputs and terminal outputs (x^1,y^h) by $\bar{S}^h$, the suprema in fixed end point problem 4.3 -(1) define a bi-function $(\bar{\phi}^h : \bar{S}^h \to]-\infty,+\infty],\ m_1 \times n_h)$. With help of the definitions in 1.5 this bi-function can be written in the form:

(1) $((\gamma_1\mu^1 + \dots + \gamma_h\mu^h) : S^1 \otimes \dots \otimes S^h \to]-\infty,+\infty],\ m_1 \times n_h)$

Next, the free end point problem van be conceived as the max-oriented dual of the bi-function in argument (2), of course provided the bi-function is proper; thus we arrive at:

$$(2)\quad ({\overset{\Delta}{}}_2(\gamma_1\mu^1+\ldots+\gamma_n\mu^h) : {\overset{\Delta}{}}_2(S^1\otimes\ldots\otimes S^h) \to \,]-\infty,+\infty],\ m_1\times n_h).$$

Another manner to apply the concept of I/O-processes and its duality properties on the dynamic max-problems is to return to the fundamental approach of 3.1 -(1) to -(6). This can be done by defining a bi-function $(\hat{\phi}^h : \hat{S}^h \to \,]-\infty,+\infty],\ (m_1+\ldots+m_h)\times(n_1+\ldots+n_h))$:

$$(3)\begin{cases} \hat{\phi}^h(x^1,a^2,\ldots,a^h;\ b^1,\ldots,b^{h-1},y^h) := \\ \quad := \sup \Sigma_{t=1}^{h}\ \gamma_t\mu^t(x^t;y^t),\ \text{over } \{x^t\}_2^h,\ \{y^t\}_1^{h-1}, \\ \qquad \text{st. (given } x^1,y^h\text{)}: (x^t,y^t)\in S^t,\ t=1,\ldots,h, \\ \qquad x^{t+1}-y^t = a^{t+1}-b^t,\ t=1,\ldots,h-1. \\ \hat{S} := \{(x^1,a^2,\ldots,a^h;\ b^1,\ldots,b^{h-1};y^h)\mid \\ \qquad \{x^t\}_2^h,\{y^t\}_1^{h-1}: (x^t,y^t)\in S^t,\ t=1,\ldots,h, \\ \qquad x^{t+1}-y^t = a^{t+1}-b^t,\ a^{t+1}\geqq 0,\ b^t\geqq 0,\ t=1,\ldots,h-1\}, \end{cases}$$

with essentially the same logic structure as the bi-function in 3.1 -(1). Analoqous to 3.2 -(2), -(3) (and using 2.10) one may arrive at a bi-function:

$$(4)\quad ((\underline{\Delta}\gamma_1\mu^1+\ldots+\underline{\Delta}\gamma_n\mu^h) : (\gamma_1\underline{\Delta}S^1)\otimes\ldots\otimes(\gamma_h\underline{\Delta}S^h) \to [-\infty,+\infty[,\ m_1\times n_h),$$

and next, by an appropriate transformation, at,what we shall call the <u>dual dynamic problem:</u>

$$(5)\begin{cases} {}^{*}\bar{\phi}^{h}(u^{1};v^{h}) := \inf \Sigma_{t=1}^{h}\gamma_t\underline{\Delta\mu}^{t}(u^{t};v^{t}), \\ \text{over } \{u^{t}\}_{2}^{h},\ \{v^{t}\}_{1}^{h-1},\ \text{st. (given } u^{1},\ v^{h}) \\ (u^{t},v^{t}) \in \underline{\Delta S}^{t},\ t = 1,\dots,h,\ (\gamma_{t+1}/\gamma_t)u_{t+1} = v_t,\ t = 1,\dots,h-1. \end{cases}$$

Clearly, a sequence $\{(u^{t},v^{t})\}_{1}^{h}$ satisfies the conditions in (5), if and only if $(\gamma_t u^{t},\gamma_t v^{t}) \in \gamma_t\underline{\Delta S}^{t}$, $t = 1,\dots,h$, $(\gamma_{t+1}u_{t+1}) = (\gamma_t v_t)$, $t = 1,\dots,h-1$; in addition one has $\gamma_t\underline{\Delta\mu}^{t}(u^{t};v^{t}) = \underline{\Delta\gamma_t\mu}^{t}(\gamma^{t}u^{t};\gamma^{t}v^{t})$, $t = 1,\dots,h$. Consequently, the set of admissible pairs (u^{1},v^{h}) of (5) is:

$$(6)\quad {}^{*}\bar{S}^{h} := \{(u^{1},v^{h}) \in R^{m_1}\times R^{n_h} \mid (\gamma_1,u^{1},\gamma_h v^{h}) \in (\gamma_1\underline{\Delta S}^{1})\otimes\dots\otimes(\gamma_h\underline{\Delta S}^{h})\}.$$

Further, the min-oriented dual of (5) in argument 1 can be written:

$$(7)\begin{cases} {}^{*}\bar{\phi}^{h}(x^{1};v^{h}) := \inf \gamma_1\langle u^{1},x^{1}\rangle + \Sigma_{t=1}^{h}\gamma_t\underline{\Delta\mu}^{t}(u^{t};v^{t}), \\ \text{over } \{u^{t}\}_{1}^{h},\ \{v^{t}\}_{1}^{h-1},\ \text{st. (given } v^{h}), \\ (u^{t},v^{t}) \in \underline{\Delta S}^{t},\ t = 1,\dots,h,\ (\gamma_{t+1}/\gamma_t)u_{t+1} = v_t,\ t = 1,\dots,h-1. \end{cases}$$

The "primal" dynamic max-problems 4.3 -(1) and -(2), and the "dual" dynamic min-problems 4.4 -(5) and -(6) are essentially symmetric with respect to each other; i.e. the same procedure, but with the opposite orientation, results, starting from the dual problem, into the original primal problems 4.3 -(1) and -(2). Consequently, taking in account the different orientation of dual problem and the coefficients (γ_{t+1}/γ_t) appearing in the equalities $(\gamma_{t+1}/\gamma_t)u_{t+1} = v_t$, $t = 1,\dots,h-1$, the properties of the primal and the dual problem are the same, of course provided the constituting I/O-processes are all lower continuous and concave.

4.5. Duality relations in the dynamic I/O-process.

Summarizing: given a dynamic I/O-process $(\mu^t: S^t \to R^1,\ m_t \times n_t)$, $t = 1,\dots,h$, and a sequence $\{\gamma_t\}_1^h \subset R^1_{++}$, we shall call $\{(u^t,v^t)\}_1^h$ a dual feasible path if:

$$(1)\qquad \begin{cases} (u^t,v^t) \in \underline{\Delta}S^t, & t = 1,\dots,h, \\ (\gamma_{t+1}/\gamma_t)u^{t+1} = v^t, & t = 1,\dots,h-1. \end{cases}$$

In this context $\{(x^t,y^t)\}_1^h$ is called a primal feasible path if:

$$(2)\qquad \begin{cases} (x^t,y^t) \in S^t, & t = 1,\dots,h, \\ x^{t+1} = y^t, & t = 1,\dots,h-1. \end{cases}$$

A sequence $\{(\tilde{u}^t,\tilde{v}^t)\}_1^h$ is called a dual strictly feasible path if it is a dual feasible path and if, in addition, a $\beta > 0$ exists so that:

$$(3)\qquad (\tilde{u}^t-\beta e,\tilde{v}^t+\beta e) \in \underline{\Delta}S^t, \quad t = 1,\dots,h,$$

e being a vector of any appropriate dimension all components equal to one. A sequence $\{(\tilde{x}^t,\tilde{y}^t)\}_1^h$ is called a primal strictly feasible path if it is a primal feasible path and if an $\alpha > 0$ exists with:

$$(4)\qquad (\tilde{x}^t-\alpha e,\tilde{y}^t+\alpha e) \in S^t, \quad t = 1,\dots,h.$$

In the context of "free-disposal" it should be clear the strictly feasibility concepts are more stringent then feasibility.

In a similar manner as in §3, the duality properties of the finite horizon process can be related to the standard duality theory. For instance, the relations between boundedness of the suprema or the infima in 4.3.-(1) and 4.4.-(5) on one side, and the existence of (strictly) feasible paths or optimal paths on the other side. We mention some special properties.

<u>4.6. Proposition</u>. Let $(\mu^t: S^t \to R^1,\ m_t \times n_t)$, $t = 1,\dots,h$ be a dynamic concave I/O-process, let $\{\gamma_t\}_1^h \subset R^1_{++}$:

(1) If $\{(\tilde{u}^t,\tilde{v}^t)\}_1^h$ is a dual strictly feasible path, then, for every primal feasible $\{(x^t,y^t)\}_1^h$ one has:

$$\gamma_h\langle\tilde{v}^h,y^h\rangle + \Sigma_{t=1}^h \gamma_t\mu^t(x^t;y^t) + $$
$$+ \beta\,\Sigma_{t=1}^h \gamma_t\langle e,x^t+y^t\rangle \leqq \gamma_1\langle\tilde{u}^1,x^1\rangle + \Sigma_{t=1}^h \gamma_t\underline{\Delta\mu}^t(\tilde{u}^t-\beta e;\tilde{v}^t+\beta e),$$

where $\beta \in R^1_{++}$ is the scalar satisfying 4.5.-(3).

(2) If $\{(\tilde{x}^t,\tilde{y}^t)\}_1^h$ is a primal strictly feasible path, then, for every dual feasible path $\{(u^t,v^t)\}_1^h$ one has:

$$\gamma_h\langle v^h,\tilde{y}^h\rangle + \Sigma_{t=1}^h \gamma_t\mu^t(\tilde{x}^t-\alpha e;\tilde{y}^t+\alpha e) + $$
$$+ \alpha\,\Sigma_{t=1}^h \gamma_t\langle e,u^t+v^t\rangle \leqq \gamma_1\langle u^1,\tilde{x}^1\rangle + \Sigma_{t=1}^h \gamma_t\underline{\Delta\mu}^t(u^t;v^t),$$

$\alpha \in R^1_{++}$ being the scalar appearing in 4.5.-(4).

(3) If $\{(\tilde{x}^t,\tilde{y}^t)\}_1^h$ and $\{(\tilde{u}^t,\tilde{v}^t)\}_1^h$ are primal and dual strictly feasible paths, then primal and dual feasible paths $\{(\hat{x}^t,\hat{y}^t)\}_1^h$, $\{(\hat{u}^t,\hat{v}^t)\}_1^h$ exist with $\hat{x}^t := \tilde{x}^1$, $\hat{v}^h := \tilde{v}^h$, such that:

$$\gamma_h\langle\tilde{v}^h,\hat{y}^h\rangle + \Sigma_{t=1}^h \gamma_t\mu^t(\hat{x}^t;\hat{y}^t) = \gamma_1\langle\hat{u}^1,\tilde{x}^1\rangle + \Sigma_{t=1}^h \gamma_t\underline{\Delta\mu}^t(\hat{u}^t,\hat{v}^t).$$

Note, given $x^1 := \tilde{x}^1$, $v^h := \tilde{v}^h$, these paths are optimal in 4.3.-(2) and 4.4.-(7) resp. Moreover one has $\tilde{\phi}^h(\tilde{x}^1;\tilde{v}^h) = {}^*\tilde{\phi}^h(\tilde{x}^1;\tilde{v}^h)$.

5. Open horizon dynamic I/O-processes.

Here we shall analyse the open horizon dynamic concave I/O-process, being composed by an infinite sequence of concave I/O-processes satisfying the conditions of 4.1. Studying the existence of ∞-horizon feasible paths, it is necessary to impose some regularity conditions on the process.

5.1. Invariant and semi-invariant dynamic I/O-processes.

We shall speak of an invariant dynamic I/O-proces if the constituting I/O-processes $(\mu^t: S^t \to R^1, m_t \times n_t)$, $t = 1,\ldots$ are equivalent. Consequently an invariant dynamic I/O-process is specified by one single I/O-process $(\mu^o: S^o \to R^n, m\times m)$.

A weaker regularity condition can be imposed by assuming that the constituting I/O-processes "converge". In order to specify this idea we define the distance of two sets $U, V \subset R^n$ as

$$(1)\quad \delta(U,V) := \inf \delta, \text{ over } \delta \in [0,+\infty], \text{st. } V \subset U + \delta\Omega,\ U \subset V + \delta\Omega,$$

where Ω is closed unit ball, and where $(+\infty)\Omega := R^n$. We say that a sequence of convex sets $\{U^t\}_1^\infty \subset R^n$ converges to $U^o \subset R^n$ if $\delta(U^o,U^t) \to 0$, for $t \to \infty$; we say that the sequence converges in the Cesàro sense if $\delta(U^o,h^{-1}\Sigma_{t=1}^h U^t) \to 0$, for $h \to \infty$.

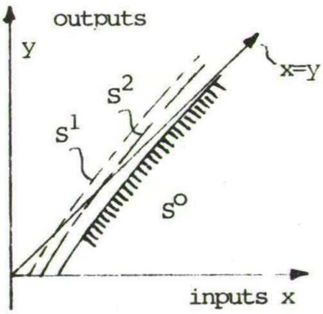
Now, using this notion, we shall call a dynamic I/O-process semi-invariant, if it is constituted by a sequence of I/O-processes $(\mu^t: S^t \to R^1, m\times m)$, $t = 1,\ldots$ where $\{\text{hypo}(S^t;\mu^t)\}_1^\infty$ converges in the Cesàro sense to $\text{hypo}(S^o;\mu^o)$, $(\mu^o: S^o \to R^1, m\times n)$ being a lower continuous concave I/O-process.

For example, a semi-invariant dynamic I/O-process appears when an invariant dynamic I/O-process $(\mu^O\colon S^O \to R^1,\ m\times n)$ is transformed by a sequence of vectors $\{r^t\}_1^\infty \subset R^m$, as described in 4.2, provided the sequence $\{h^{-1}\Sigma_{t=1}^h r^t\}_1^\infty$ converges.

<u>5.2. Proposition.</u> Let $(\mu^t\colon S^t \to R^1,\ m\times m)$, $t = 1,2,\ldots$ be a sequence of concave I/O-processes which converges in the Cesàro sense to the dynamic I/O-process $(\mu^O\colon S^O \to R^1,\ m\times m)$. Let $\rho \in R^1_+$. Suppose there exists an $\{(x^t,y^t)\}_1^\infty$ satisfying: $(x^t,y^t) \in S^t$, $\rho x^{t+1} < y^t$, $t = 1,2,\ldots$
Then, for every $\varepsilon > 0$, there is a $(\bar{x}^\varepsilon,\bar{y}^\varepsilon) \in S^O+\varepsilon\Omega$ so that $\rho\bar{x}^\varepsilon < \bar{y}^\varepsilon$.

<u>Proof.</u> Concerning $\{(x^t,y^t)\}_1^\infty$ as assumed above, one has:
$(1/h)\rho\Sigma_{t=1}^h x^t < (1/h)\rho x^1 + (1/h)\Sigma_{t=1}^h y^t$, $h = 1,2,\ldots,$
$(1/h)\Sigma_{t=1}^h (x^t,y^t) \in (1/h)\Sigma_{t=1}^h S^t$, $h = 1,2,\ldots$.
the first relation being a consequence of $S^t \subset R^m_+\times R^m_+$, $t = 1,2,\ldots$.
Next, Cesàro convergency implies that for every $\varepsilon > 0$ an h exists so that $(1/h)\Sigma_{t=1}^h (x^t,y^t) \in S^O + \frac{1}{2}\varepsilon\Omega$, $(1/h)\rho\,\|x^1\| \leqq \frac{1}{2}\varepsilon$, which proves the theorem. □

The picture gives an example $S^O \subset R^1_+\times R^1_+$, $\rho := 1$, where no combination $(x,y) \in S^O$ exists with $x < y$.



Of course, with respect to invariant dynamic concave I/O-processes a stronger result may be deduced. For instance:

5.3. Proposition. Let (μ: S $\to R^1$, m×m) be a concave I/O-process, let $\rho \in R^1_+$. Then the following statements are equivalent:

(1) $\forall\, c \in R^m_{++}: \exists\{(x^t,y^t)\}_1^\infty \subset S: \rho x^{t+1} \leqq y^t - c,\ t = 1,2,\ldots$

(2) $\exists (x^o,y^o) \in S: \rho x^o < y^o$.

Proof. That (2) is an implication of (1) may be verfied in a similar manner manner as 5.2. The other way round: if $(x^o,y^o) \in S$ satisfies $\rho x^o < y^o$, then $\{(x^t,y^t)\}_1^\infty$, $(x^t,y^t) := (x^o,y^o)$, $t = 1,2,\ldots$, $c := y^o - \rho x^o$ satisfy the condition in (1). □

Observe that, of course, these proposition are exlusively related to the domain of the bi-function representing the I/O-structure; for instance putting the function values idential to zero, the character of proposition 5.2 and 5.3 will not change.

An other interesting question is the boundedness of sequences constituting a feasible path. We start with the following auxiliary proposition.

5.4. Proposition. Let (μ:S $\to R^1$,m×m) be a lower continuous concave I/O-process. Suppose $\rho \in R^1_{++}$, $\delta \in R^1$ are numbers so that the set $Q := \{(x,y) \in S \mid \rho x = y,\ \mu(x;y) \geqq \delta\} \neq \emptyset$. Then the set Q is bounded if, and only if, there is a pair $u,v \in R^m$, $(u,v) \in \underline{\Delta}\, S$, $u < \rho v$.

Proof. Let $(u,v) \in \underline{\Delta}\, S$, $u,v \in R^m$. Then, for every $(x,y) \in Q$, one has: $\delta \leqq \mu(x;y) \leqq \underline{\Delta}\, \mu(u;v) - \langle \rho v-u,x\rangle$. With $u < \rho v$, and with $\rho x=y$, $x,y \geqq 0$ for all $(x,y) \in Q$, the latter implies boundedness of Q. Thus the existence of an $(u,v) \in \underline{\Delta}\, S$, $\rho u < v$, implies boundedness of Q.

The other way round, let $A := \{(u,v) \in R^m \times R^m | u = \rho v\}$. Then the assumption that there is no $(u,v) \in \underline{\Delta}\, S$, $u < \rho v$, implies $A \cap \operatorname{rint}(\underline{\Delta} S) = \emptyset$. Since $\underline{\Delta}\, S$ is a convex set, the geometric Hahn-Banach theorem affirmates the existence of a $(p,q) \in R^m \times R^m$ such that :

- $\forall (u,v) \in A$: $\langle p,u\rangle + \langle q,v\rangle = 0$,
- $\forall (u,v) \in \underline{\Delta} S$: $\langle p,u\rangle + \langle q,v\rangle \leqq 0$.

Clearly, the first relation implies $p = -\rho q$; the second property implies that for every $(x,y) \in \underline{\nabla}\, \underline{\Delta}\, S$, $(u,v) \in \underline{\Delta} S$, and every $\lambda \geqq 0$:
$\underline{\Delta}\mu(u;v) + \langle x+\lambda\alpha p,u\rangle - \langle y+\lambda q,v\rangle \geqq \underline{\nabla}\underline{\Delta}\mu(x;y)$. Since by lower continuity and concavity of the I/0-process $\underline{\nabla}\underline{\Delta}\mu : \underline{\nabla}\underline{\Delta}\, S \to R^1$ is identical to $\mu : S \to R^1$, the latter implies that Q is not bounded, indeed. □

Observe that the proposition includes also a statement concerning "I/0-sets"; for putting f identical to zero, the proposition implies:
$Q := \{(x,y) \in S | \rho x = y\}$ (provided $Q \neq \emptyset$) is bounded if and only if there is a $u,v \in R^m$, $\rho u < v$ with the property that $\{\beta \in R^1 | f(x,y) \in S$: $\beta = -\langle u,x\rangle + \langle v,y\rangle\}$ has an opper bound. A similar observation can be made with respect to the following proposition.

<u>5.5. Proposition.</u> Let $(\mu : S \to R^1, m{\times}m)$ be a lower continuous concave I/0-process. Let $\rho \in R^1_{++}$, $\delta \in R^1$. Suppose that the set
$Q := \{(x,y) \in S | \rho x = y,\ \mu(x;y) \geqq \delta\}$ is not empty.
Then boundedness of Q implies that every $\{(x^t,y^t)\}^\infty \subset S$, satisfying $\rho x^{t+1} = y^t$, $t = 1,\ldots$ and $\mu(x^t;y^t) \geqq \nu$, $t = 1,\ldots$ for any $\nu \in R^1$ is bounded as well.

<u>Proof</u>. Let $\{(x^t,y^t)\}_1^\infty$ satisfy the conditions. Let $\alpha \in\,]0,1[$ and define

$\{\beta^h\}_1^\infty$ by $\beta := (1-\alpha)/(1-(\alpha)^h)$. Define $\{\bar{x}^h\}_1^\infty$, $\{\bar{y}^h\}_1^\infty$ by :
$\bar{x}^h := \beta^h \Sigma_{t=1}^h (\alpha)^{h-t} x^t$ and $\bar{y}^h := \beta^h \Sigma_{t=1}^h (\alpha)^{h-t} y^t$, $h = 1, \ldots$, then by non-neg. of $\{x^t\}_1^\infty$, $\{y^t\}_1^\infty$ one has:

(1) $x^h \in [0,\bar{x}^h]$, $y^h \in [0,\bar{y}^h]$, $h = 1, 2, \ldots$

Since $\beta^h \Sigma_{t=1}^h (\alpha)^{h-t}(x^t,y^t)$ is a convex combination, concavity of $\mu:S \to R^1$ implies $(\bar{x}^h, \bar{y}^h) \in S$, $\mu(\bar{x}^h;\bar{y}^h) \geqq \nu$, $h = 1, \ldots$. Non-negativity of $\{y^t\}_1^\infty$ and of x^1, and $\alpha \in]0,1[$ implies that $\rho\bar{x}^h \leqq \alpha\bar{y}h + \rho x^1$, $h = 1,2,\ldots$

Thus, under assumption 1.1-(2) (i.e. "free disposal") one may conclude:

(2) $\{(\bar{x}^h,\bar{y}^h)\}_1^\infty \subset \bar{Q}_\alpha := \{(x,y) \in S | \rho x = \alpha y + x^1, \mu(x;y) \geqq \nu\}$. Clearly, by (1) and (2), boundedness of $\bar{Q}_\alpha$ implies boundedness of the sequence $\{(x^t,y^t)\}_1^\infty$.

In order to prove that $\bar{Q}_\alpha$ is bounded, we shall use the concept recession cone (cf. Rockefellar [11]), with respect to a closed convex set $C \subset R^n$, being defined by:

rec$(C) := \{y \in R^n | \quad x \in C: \quad \lambda \in R_+^1 : x + \lambda y \in C\}$.

By virtue of the property that a closed convex set is bounded if and only if its recession cone is equal to $\{0\}$, boundedness of Q implies:

$\forall w \in R^m \neq 0: (\rho w,w,0) \notin$ rec(hypo$(S;\mu)$). Since the recession cone of a convex set is a closed convex cone the latter implies the existence of a $\gamma \in]0,1[$ (close enough to 1) such that:

$\forall w \in R^m \neq 0$: $(\rho w,\gamma w,0) \notin$ rec(hypo$(S;\mu)$).

Consequently, setting $\alpha := \gamma$, the definition of the set $\bar{Q}_\alpha$ implies:

(3) rec$(\bar{Q}_\alpha) = \{0\}$. ($\alpha \in]0,1[$, close enough to 1).

Clearly, (1), (2), and (3) prove the proposition. □

Putting 5.5. in the opposite direction, we have :

5.6. Proposition. Let (μ: $S \to R^1$,m×m) be lower continuous concave I/O--process. Suppose $\{(\tilde{x}^t,\tilde{y}^t)\}_1^\infty \subset S$ satisfies $\rho\tilde{x}^{t+1} = y^t - c$, $\mu(\tilde{x}^t;\tilde{y}^t) \geqq \nu$, $t = 1,\ldots,$ for some $c \in R^m_{++}$, $\rho \in R^1_{++}$, $\nu \in R^1$. If the set $Q := \{(x,y)S|\rho x = y,\ \mu(x;y) \geqq \delta\}$ (δ being any number so that $Q \neq \emptyset$) is not bounded, then there is a $z \in R^m_+$, $\neq \emptyset$ such that $\{(\tilde{x}^t+tz,\tilde{y}^t+tz)\}_1^\infty \subset S$, $\rho(\tilde{x}^{t+1}+tz+z) \leqq \tilde{y}^t+tz$, $\mu(\tilde{x}^t+tz;\tilde{y}^t+tz) \geqq \nu$, $t = 1,\ldots$

Proof. If Q is not bounded, then the recession cone of Q is not equal to $\{0\}$. Hence, there is a $(\bar{x},\bar{y})$, $\rho\bar{x} = \bar{y} \neq 0$ so that, for every $(x,y) \in S$, $\lambda \in R^1_+$: $(x+\lambda\bar{x},y+\lambda\bar{z}) \in S$, $\mu(x+\lambda\bar{x};y+\lambda\bar{z}) \geqq \mu(x;y)$. Now putting $z := \lambda\bar{y}$, λ positive and such that $\lambda\rho z \leqq c$, it should be clear that z satisfies the conditions. □

5.7. Corollary of 5.3. to 5.6. Let (μ:$S \to R^1$,m×m) be a lower continuous concave I/O-process. Let $\rho \in R^1_{++}$, and suppose there is a sequence $\{(\tilde{x}^t,\tilde{y}^t)\}_1^\infty \subset S$ satisfying $\rho\tilde{x}^{t+1} = y^t-c$, $\mu(\tilde{x}^t;\tilde{y}) \geqq \nu$, $t = 1,\ldots,$ for some $c \in R^m_{++}$, $\nu \in R^1$. Then the following statements are equivalent :

(1) All sequences $\{(x^t,y^t)\}_1^\infty \subset S$ satisfying $\rho x^{t+1} = y^t$, $t = 1,\ldots$ and $\mu(x^t;y^t) \geqq \alpha$, $t = 1,\ldots$ for some $\alpha \in R^1$, are bounded.

(2) All sets $Q_\nu := \{(x,y) \in S|\rho x = y,\ \mu(x;y) \geqq \nu\}$ are bounded.

(3) There exist an $(u,v) \in \underline{\Delta}\ S$ so that $u < \rho v$.

(4) There exist an $\{(u^t,v^t)\}_1^\infty \subset \underline{\Delta}\ S$, $b \in R^m_+$ so that $u^{t+1} \leqq \rho v^t - b$, $t = 1,\ldots$.

Proof. The equivalence of (1) and (2) is the consequence of 5.5. and 5.6. The equivalence of (2) and (3) is affirmated by 5.4. The equivalence of (3) and (4) can be demonstrated by applying 5.3. on the dual process $(\underline{\Delta}\mu:\underline{\Delta}S \to R^1,\ m\times m)$. □

Thus it appears that boundedness under a lower bound condition for the function values is directly related to the existence of a feasible path of the "dual process". In 5.9. the duality aspects will be elaborated on by studying optimality aspects.

<u>5.8. The optimality criterion.</u> Let $(\mu^t:S^t \to R^1,\ m_t\times n_t), t = 1,2,\ldots$ be an open dynamic I/0-process. As in the finite horizon case we shall restrict ourselves to the "weighted-sum" objective function. The natural way is to consider the limit $\lim_{h\to\infty} \Sigma_{t=1}^{h}\gamma_t\mu^t(x^t;y^t)$, with $\{\gamma_t\}_1^\infty \subset R^1_{++}$. However, such a criterion fails in the case that the sequence $\{\Sigma_{t=1}^{h}\gamma_t\mu^t(x^t;y^t)\}_{h=1}^\infty$ does not converge. Therefore we transfer the criterion used by Halkin [7] and others to our problem; i.e. given the initial state $\tilde{x}^1$, we say that $\{\hat{x}^t,\hat{y}^t)\}_1^\infty$ is an <u>optimal path</u> if it is feasible with $\hat{x}^1 := \tilde{x}^1$, and if there does not exist a feasible path $\{(x^t,y^t)\}_1^\infty$, $x^1 := \tilde{x}^1$, such that for some $\epsilon > 0$ and some positive integer $T > 0$:

$$(1)\quad \Sigma_{t=1}^{h}\gamma_t\mu^t(x^t;y^t) \geqq \epsilon + \Sigma_{t=1}^{h}\gamma_t\mu^t(\hat{x}^t;\hat{y}^t),\ h = T, T+1,\ldots .$$

In words: given the initial state, a feasible path is called optimal if there does not exist a feasible path which is "in the long run substantially better". Comparing feasible paths $\{(\hat{x}^t,\hat{y}^t)\}_1^\infty$ and $\{(x^t,y^t)\}_1^\infty$ with $\hat{x}^1 = x^1$, with converging sequences $\{\Sigma_{t=1}^{h}\gamma_t\mu^t(\hat{x}^t;\hat{y}^t)\}_{h=1}^\infty$, $\{\Sigma_{t=1}^{h}\gamma_t\mu^t(x^t;y^t)\}_{h=1}^\infty$ it is clear that the first is "better" in the sense of (1) if, and only if :

$\lim_{h\to\infty}(\Sigma_{t=1}^{h}\gamma_t\mu^t(\hat{x}^t;\hat{y}^t) > \lim_{h\to\infty}\Sigma_{t=1}^{h}\gamma_t\mu^t(x^t;y^t)$. Thus in case of convergency the "limit-criterion" is equivalent with our "no better" criterion.

In the context of (1), we say, that $\{(\hat{x}^t,\hat{y}^t)\}_1^\infty$ is dominated by $\{(x^t,y^t)\}_1^\infty$ if one has the inequalities for some $\varepsilon > 0$, $T \in \{1,2,...\}$.

5.9. Dual open-horizon I/0-processes. The natural way to study the open-horizon max-problem of 5.8. is to treat the system as a sequence of shifting finite horizon problems, being described in 4.3-(1) or 4.3-(2). The corresponding system of dual problems defined by 4.4-(6) or -(7), of course, induces an open horizon dual problem with the same, but minimal oriented, optimality criterion. Starting from the dynamic I/0-process $(\mu^t:S^t \to R^1, m_t \times n_t)$, $t = 1,2,...$ and the sequence $\{\gamma_t\}_1^\infty \subset R^1_{++}$, we condense the max-oriented primal problem and the min-oriented dual problem infinite horizon problem in the notations:

(1) $\sup_{h\to\infty}\Sigma_{t=1}^{\infty}\gamma_t\mu^t(x^t,y^t)$, over $\{x^t\}_2^\infty$, $\{y^t\}_1^\infty$,
st. (given x^1) $(x^t,y^t)t\ S^t$, $x^{t+1} = y^t$, $t = 1,...$

(2) $\inf_{h\to\infty}\gamma_1\langle x^1,u^1\rangle + \Sigma_{t=1}^{\infty}\gamma_t \underline{\Delta}\ \mu^t(u^t;v^t)$, over $\{u^t\}_1^\infty$, $\{v^t\}_1^\infty$
st. $(u^t,v^t) \in \underline{\Delta}\ S^t$, $(\gamma_{t+1}/\gamma_t)u^{t+1} = v^t$, $t = 1,...$.

For the finite horizon subproblems we refer to 4.3-(1) and 4.4-(6). The concepts primal and dual feasible solutions shall be used in the same manner as in 4.5. With respect to strictly feasible paths we impose an extra condition, which is relevant only in the ∞-horizon context: A primal feasible path $\{\tilde{x}^t,\tilde{y}^t)\}_1^\infty$ is called primal strictly feasible if the following conditions are satisfied:

$$(3)\quad \begin{cases} \{(\tilde{x}^t,\tilde{y}^t)\}_1^\infty \text{ and } \{\mu^t(\tilde{x}^t;\tilde{y}^t;\tilde{y}^t)\}_1^\infty \text{ are bounded,} \\ \exists\, \alpha \in R^1_{++}:\ (\tilde{x}^t-\alpha e,\ \tilde{y}^t+\alpha e) \in S^t,\ t=1,\ldots \end{cases}$$

A dual feasible path $\{(\tilde{u}^t,\tilde{v}^t)\}_1^\infty$ is called <u>dual strictly feasible if</u>:

$$(4)\quad \begin{cases} \{(\tilde{u}^t,\tilde{v}^t)\}_1^\infty \text{ and } \{\underline{\Delta}\,\mu^t(\tilde{u}^t;\tilde{v}^t)\}_1^\infty \text{ are bounded,} \\ \exists\, \beta \in R^1_{++}:\ (\tilde{u}^t-\beta e,\ \tilde{v}^t+\beta e) \in \underline{\Delta}\, S^t,\ t=1,\ldots\ . \end{cases}$$

The open-horizon process is called primal (resp. dual) (strictly) if there exists a primal (resp. dual) (strictly) feasible solution. In the case that the underlying dynamic I/O-process is invariant and concave, proposition 5.3 shows that the problem is primal strictly feasible if, and only if, there is an $(x,y)\in S$ with $x < y$.
If one has an exponential sequence of weighting factors: $\gamma_t := (\pi)^t$, $\pi \in R^1_{++}$ being the <u>time-discount factor</u>, the equality constraints in the dual problem reduce to $\pi u^{t+1} = v^t$, $t = 1,\ldots$ If, in addition the process is invariant, then, applying 5.3. on the dual I/O-process, it appears that the problem is dual strictly feasible if and only if there is an $(u,v) \in \underline{\Delta}\, S$ with $\pi u < v$.

As a fundamental starting point in deducing necessary and sufficient optimality conditions we have the following inequalities concerning primal and dual feasible paths $\{(x^t,y^t)\}_1^\infty$, $\{(u^t,v^t)\}_1^\infty$:

$$(4)\quad \gamma_h \langle v^h,y^h\rangle + \Sigma_{t=1}^h \gamma_t \mu^t(x^t;y^t) \leqq \gamma_1\langle u^1,x^1\rangle + \Sigma_{t=1}^h \gamma_t \underline{\Delta}\mu^t(u^t;v^t),$$

for all $h \in \{1,2,\ldots\}$. Since each $\gamma_h\langle v^h,y^h\rangle$ is non-negative, the optimality criterion of 5.8. and these inequalities imply :

5.10. Proposition. (sufficient optimality conditions). If one has primal and dual feasible paths $\{(\hat{x}^t,\hat{y}^t)\}_1^\infty$, $\{(\hat{u}^t,\hat{v}^t)\}_1^\infty$ such that simultaneously:

(1) $\forall h \in \{1,2,\ldots\}$:

$$\gamma_h <\hat{v}^h,\hat{y}^h>+\Sigma_{t=1}^h \gamma_t \mu^t(\hat{x}^t;\hat{y}^t) = \gamma_1 <\hat{u}^1,\hat{x}^1>+\Sigma_{t=1}^h \gamma_t \underline{\Delta}\, \mu^t(\hat{u}^t;\hat{v}^t),$$

(2) $\gamma_h <\hat{v}^h,\hat{y}^h> \to 0$, if $h \to \infty$,

then, for a fixed $x^1 := \hat{x}^1$, these paths are optimal in 5.9-(1) and 5.9-(2) resp.

Elsewhere we proved (if. [2], [4]) that, in the case primal and dual strictly feasible exist, these conditions are necessary for optimality, as well. In addition it is shown that there exists a primal and a dual optimal solution. Special optimal solutions satisfying 5.10-(1), -(2) are discussed in §6.

Another useful apparatus is given by the inequalities in 4.6-(1) and -(2), concerning strictly feasible paths. For instance, with help of these it can be verified that a necessary condition for the process to be simultaneously primal and dual strictly feasible is boundedness of the sequence $\{\Sigma_{t=1}^h \gamma_t\}_{h=1}^\infty$. With respect to an exponential sequence $\gamma_t := (\pi)^t$, $t = 1,\ldots$ this condition implies that $\pi \in]0,1[$.

6. Invariant optimal paths in open dynamic I/O-processes.

Starting from an invariant open dynamic I/O-process and exponential weighted object function, we shall proof the existence of primal and dual optimal solution which are invariant over the periods. Formally, we consider the following system.

6.1. The invariant open dynamic I/O-process and invariant paths.

Let $(\mu\colon S \to R^1,\ m\times m)$ be an I/O-process and let $\pi \in]0,1[$ be a time-discount factor, defining the ∞-horizon problem:

$$\begin{array}{ll} (1) & \sup_{h\to\infty} \Sigma_{t=1}^{h} (\pi)^t \mu(x^t;y^t), \quad \text{over } \{x^t\}_2^\infty,\ \{y^t\}_1^\infty, \\ & \text{st. (given } x^1) \quad (x^t,y^t) \in S,\ \ x^{t+1} = y^t,\ \ t = 1,\ldots, \end{array}$$

$$\begin{array}{ll} (2) & \inf_{h\to\infty} \pi\langle x^1,u^1\rangle + \Sigma_{t=1}^{h} (\pi)^t \underline{\Delta}\mu(u^t;v^t), \quad \text{over } \{u^t\}_1^\infty,\ \{v^t\}_1^\infty, \\ & \text{st. (given } x^1) \quad (u^t,v^t) \in \underline{\Delta}S,\ \ \pi u^{t+1} = v^t,\ \ t = 1,\ldots \end{array}$$

Clearly, this is exactly the invariant version of the open-horizon problem 5.5.-(1), -(2) with $\gamma_t := (\pi)^t$, $t = 1,\ldots$

We call $((\hat{x},\hat{y}),(\hat{u},\hat{v}))$ an <u>invariant optimal combination</u> if:

$$(3)\quad \begin{cases} (\hat{x},\hat{y}) \in S, & \hat{x} = \hat{y} \\ (\hat{u},\hat{v}) \in \underline{\Delta}S, & \pi\hat{u} = \hat{v} \\ \langle\hat{v},\hat{y}\rangle + \mu(\hat{x};\hat{y}) = \langle\hat{u},\hat{x}\rangle + \underline{\Delta}\mu(\hat{u};\hat{v}). \end{cases}$$

Putting the fixed initial vector in (1) and (2) equal to $\hat{x}$, the condition implies that the sequences $x^{t+1} := \hat{x}$, $y^t := \hat{y}$, $t = 1,\ldots$ and $(u^t,v^t) := (\hat{u},\hat{v})$, $t = 1,\ldots$ are primal and dual feasible. In addition, since $\pi \in\,]0,1[$ and since $\gamma_t := (\pi)^t$, $t = 1,\ldots,$ it appears that these paths satisfy the sufficient conditions for optimality, given in 5.10. Thus, proving the existence of invariant optimal combinations, we prove the existence of optimal invariant paths.

The definition of invariant optimal combinations correspond with an interesting economic interpretation. Namely, taking the dual part in the combination as input/output prices, one has the net value maximization problem:

$$(4) \qquad \sup\ \mu(x;y) - \langle\hat{u},x\rangle + \pi\langle\hat{u},y\rangle \quad \text{over } (x,y) \in S,$$

which delivers the primal part $(\hat{x},\hat{y})$ as an optimal solution. The input and output prices are the same; however, the value of the outputs $\langle\hat{u},y\rangle$ is discounted by the time discount factor π. This looks very reasonable, for the value of the outputs is realized one period later than the input costs are made.

<u>6.2. Determining invariant optimal combinations as a parametric programming problem</u>. Let us consider the max-problems:

$$(1) \quad \begin{cases} \theta(z) := \sup\ \mu(x;y), \quad \text{over } (x,y) \in S, \quad \text{st. } x-\pi y = (1-\pi)z, \quad \text{where} \\ z \in Z := \{\tilde{z} \in R^m \mid \exists(\tilde{x},\tilde{y}) \in S\colon \tilde{x}-\pi\tilde{y} = (1-\pi)\tilde{z}\}, \end{cases}$$

together with the corresponding Lagrangian representations $L_z\colon S{\times}R^m \to R^1$:

(2) $L_z(x,y;w) := \mu(x;y)-\langle w,x-\pi y\rangle+(1-\pi)\langle w,z\rangle.$

With respect to any saddle point $(\hat{x},\hat{y}) \in S$, $\hat{w} \in R^m$, given $z \in Z$ being defined by the condition that, for all $(x,y) \in S$, $w \in R^m$:

(3) $L_z(x,y;\hat{w}) \leqq L_z(\hat{x},\hat{y};\hat{w}) \leqq L_z(\hat{x},\hat{y};w)$,

the first inequality and the definition relation $\underline{\Delta}\mu(\hat{w};\pi\hat{w}) :=$
$:= \sup \mu(x;y)-\langle\hat{w},x\rangle+\langle\pi\hat{w},y\rangle$, over $(x,y) \in S$, imply:

(4) $\begin{cases} (\hat{w},\pi\hat{w}) \in \underline{\Delta}S \\ L_z(\hat{x},\hat{y};\hat{w}) = (1-\pi)\langle\hat{w},z\rangle+\underline{\Delta}\mu(\hat{w};\pi\hat{w}). \end{cases}$

The second inequality implies:

(5) $\hat{x}-\pi\hat{y} = (1-\pi)z.$

Now, suppose we have a $\hat{z} \in Z$ and a corresponding saddle point $(\hat{x},\hat{y}),\hat{w}$ such that $\hat{y} = \hat{z}$. Then (5) implies that $\hat{x} = \hat{y}$. Relation (4), definition (2) and the definition of $\underline{\Delta}\mu(\hat{w};\pi\hat{w})$ imply $\langle\pi\hat{w},\hat{y}\rangle+\mu(\hat{x};\hat{y}) = \langle\hat{w},\hat{z}\rangle+\underline{\Delta}\mu(\hat{u};\hat{w})$. Thus, putting $\hat{u} := \hat{w}$, $\hat{v} := \pi\hat{w}$, an invariant optimal combination is identified. Summarizing:

<u>6.3. Proposition</u>. If, given $z := \hat{z}$, the triple $(\hat{x},\hat{y},\hat{w})$ is a saddle point of the Lagrangian representation 6.2.-(2), such that $\hat{y} = \hat{z}$, then $((\hat{x},\hat{y}),(\hat{u},\hat{v}))$ with $\hat{u} := \hat{w}$, $\hat{v} := \pi\hat{w}$, is an invariant optimal combination.

6.4. Invariant optimal solutions as a Kakutani-fixed point.

As a consequence of 6.2 and 6.3 invariant optimal combinations can be identified as solutions of a fixed point problem. Namely, let $Y: Z \to R^m$ be a multi-function deduced from $\theta: Z \to]-\infty,+\infty]$ (cf. 6.2.-(1)) by:

$$(1) \quad Y(z) := \{y \in R^m \mid \exists x \in R^m: (x,y) \in S, \ x-\pi y = (1-\pi)z, \ \mu(x;y) = \theta(z)\}.$$

Suppose, $\hat{z}$ is a fixed point of $Y: Z \to R^1$ (i.e. $\hat{z} \in Z$, $\hat{z} \in Y(\hat{z})$), and suppose that $\hat{w}$ is a Lagrangian vector with respect to $\hat{z}$. Then, one may verify that $(\hat{x},\hat{y},\hat{w})$ with $\hat{y} := \hat{z}$, $\hat{x} := \hat{z}$ exactly is a saddle point of 6.2.-(2), given $z := \hat{z}$. And next, by 6.3, we may conclude that an invariant optimal combination is identified.

Under some assumptions we shall prove the existence of fixed points with the help of Kakutani's fixed point theorem. There, the existence of a fixed point of $Y: Z \to R^m$ is ensured if a subset $Z^o \subset Z$ can be found such that

- Z^o is convex and compact,
- the sets $Y(z)$ are convex for all $z \in Z^o$,
- $\text{hypo}(Z^o;Y) := \{(z,y) \in Z^o \times R^m \mid y \subset Y(z)\}$ is closed,
- $Z^o \subset Y(Z^o)$.

Assume, we have a pair $(\tilde{u},\tilde{v}) \in \underline{\Delta}S$ satisfying $\tilde{u} < \tilde{v}$. Then we also have a $\delta > 1$, an $(\bar{u},\bar{v}) \in \underline{\Delta}S$ so that $0 < \bar{u}$, $\delta\bar{u} \leqq \bar{v}$. Elaborating the inequelity $\mu(x;y)-\langle\bar{u},x\rangle+\langle\delta\bar{u},y\rangle \leqq \underline{\Delta}\mu(\bar{u};\bar{v})$, one will find:

$$(2)\quad \begin{cases} \forall (x,y) \in S, \quad z \in R^m \mid x - \pi y = (1-\pi)z: \\ \mu(x;y) \leqq \Delta\mu(\bar{u};\bar{v}) + \langle (1-\pi)\bar{u}, z\rangle - \langle (\delta-\pi)\bar{u}, y\rangle . \end{cases}$$

Next, assume that we have an $(\tilde{x},\tilde{y}) \in S$ so that $\pi\tilde{x} < \tilde{y}$. Then, provided $z \geqq 0$, a necessary condition for $(x,y) \in S$, to be optimal in max-problem 6.2.-(1), is the inequality $\mu(\tilde{x};\tilde{y}) \leqq \mu(x;y)$. In connection with (2), this condition implies the existence of a $\beta \in R^1$ so that, for every $z \in R^m_+$, the inequality

$$(3)\quad \langle (\delta-\pi)\bar{u}, y\rangle \leqq \beta + \langle (1-\pi)\bar{u}, z\rangle$$

is a necessary condition for an $(x,y) \in S$ to be optimal in 6.2.-(1). Now let $\gamma := \beta/(\delta-1)$, and let $Z^o := \{z \in R^m_+ \mid \langle \bar{u}, z\rangle \leqq \gamma\}$. Then the necessary condition for optimality (3) imples that for every $z \in Z^o$, the inequality

$$(4)\quad \langle \bar{u}, y\rangle \leqq \gamma,$$

is a necessary condition for optimality. Assuming that $\mu: S \to R^1$ is lower continuous, we also have that for every $z \in Z^o$: $Y(z) \neq \emptyset$. Thus we have

$$(5)\quad Y(Z^o) \subset Z^o.$$

Further the set Z^o is compact and convex. Closedness of hypo$(Z^o;Y)$ is a well known continuity property in the optimality theory (cf. Berge: Espaces Topologiques 1959). Finally, concavety of $\mu: S \to R^1$ implies convexity of the sets $Y(z)$. Hence, by virtue of Kakutani's fixed point theorem, the assumptions imply the existence of a fixed point.

Concerning the dual side, the assumption that there is a $(\tilde{x},\tilde{y}) \in S$, $\pi\tilde{x} < \tilde{y}$ implies, for every $z \in R^m_+$, the existence of a Lagrange vector in 6.2.-(1). Thus:

<u>6.5. Proposition</u>. Suppose $(\mu: S \to R^1, \ m\times m)$ is an lower continuous concave I/O-process. If there is a $(\tilde{x},\tilde{y}) \in S$ and a $(\tilde{u},v) \in \underline{\Delta}S$ so that $\pi\tilde{x} < \tilde{y}$, $\tilde{u} < v$, then there exists an invariant optimal combination.

It is interesting to relate these condition to the results concerning the existence and boundedness of feasible pathsin invariant dynamic I/O-processes.

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List of simbols.

R^n n-dimensional real vector space

$R^n_+ := \{x \in R^n \mid x_i \geqq 0, \quad i = 1,\dots,n\}$

$R^n_{++} := \{x \in R^n \mid x_i > 0, \quad i = 1,\dots,n\}$

$\|\ \|$ the euclidean norm

$\langle x,u\rangle$ the inner product of vectors $x,u \in R^n$

$[x,y] := \{\lambda x+(1-\lambda)y \mid 0 \leqq \lambda \leqq 1\}$

$]x,y] := \{\lambda x+(1-\lambda)y \mid 0 < \lambda \leqq 1\}$

${}^+(z)$ given $z \in R^n$, ${}^+(z)_i := z_i$ if $z_i \geqq 0$, else ${}^+(z)_i := 0$

${}^-(z)$, given $z \in R^n$, ${}^-(z_i) := {}^-z_i$ if $z_i \leqq 0$, else ${}^-(z)_i := 0$

e finite dimensional vector all components equal to 1

cl(S), the closure of a set S

rint(S), the relative interior of a set S

CONVEX PROCESSES AND HAMILTONIAN DYNAMICAL SYSTEMS

by

R.T. Rockafellar

January, 1978

CONVEX PROCESSES AND HAMILTONIAN DYNAMICAL SYSTEMS

R.T. Rockafellar*)

Many economists have studied optimal growth models of the form

$$\text{(1)}\quad \begin{aligned}&\text{maximize } \int_0^\infty e^{-\rho t}\, U(k(t),z(t))dt\\ &\text{subject to } k(0) = k_0,\ k(t) = z(t) - \gamma k(t),\end{aligned}$$

where k is a vector of capital goods, γ is the rate of depreciation, ρ is the discount rate, and U is a continuous concave utility function defined on a closed convex set D in which the pair (k,z) is constrained to lie. The theory of such problems is plagued by technical difficulties caused by the infinite time interval. The optimality conditions are still not well understood, and there are serious questions about the existence of solutions and even the meaningfulness, in certain cases, of the expression being maximized.

One thing is clear, however. Any trajectory k(t) which is worthy of consideration as optimal in (1) would in particular have to have the property that for every finite time interval $[t_0,t_1] \subset [0,\infty)$ one has

$$\text{(2)}\qquad \int_{t_0}^{t_1} e^{-\rho t}\, U(k(t),k(t)+\gamma k(t))dt \le \int_{t_0}^{t_1} e^{-\rho t}\, U(\bar{k}(t),\bar{k}(t)+\gamma\bar{k}(t))dt.$$

(For otherwise, the portion of k over $[t_0,t_1]$ could be replaced by k, and this would constitute a definite improvement.) This condition severely limits candidates for optimal paths and allows us to study them in terms of Hamiltonian dynamical systems involving subgradients.

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Hamiltonian dynamical systems arise in the optimality conditions for variational problems of the form

$$\text{(3)}\qquad \begin{aligned} &\text{minimize} \int_{t_0}^{t_1} L(t,x(t),\dot{x}(t))dt \\ &\text{subject to } x(t_0) \in D_0,\ x_1(t_1) \in D_1. \end{aligned}$$

Classically, one always supposed L to be a finite, differentiable function, but for the purpose of applications to economic models it is essential that one be able to treat the case where $L(t,\cdot,\cdot)$ is for each t a closed, proper, convex function on $R^n \times R^n$. The theory of problem (3) had been extended in this direction by Rockafellar [1], [2], [3]. The model (1) corresponds with the change of notation $x(t) = e^{\gamma t}k(t)$ to

$$\text{(4)}\qquad L(t,x,v) = -e^{-\rho t}\, U(e^{-\gamma t}x, e^{-\gamma t}v),$$

where U is interpreted as $-\infty$ outside of D.

Of course something must be assumed about the way that L depends on t. The correct condition in general is that L should be a "normal integrand" [1], [4]. This technical property of measurability will not be discussed here, but it is certainly satisfied when L is of the form (4) (under the assumptions already stated) and also when L is independent of t. Concerning the trajectory x(t), one does not have to assume differentiability, but merely absolute continuity; the time derivative $\dot{x}(t)$ then exists for almost every t.

The <u>Hamiltonian</u> associated with L is the function

$$\text{(5)}\qquad H(t,x,p) = \sup_{v \in R^n}\{p.v - L(t,x,v)\}.$$

Thus $H(t,x,\cdot)$ is the convex function conjugate to $L(t,x,\cdot)$, so that L is in turn determined uniquely by H:

$$L(t,x,v) = \sup_{p \in R^n}\{p.v - H(t,x,p)\}.$$

Since $L(t,x,v)$ is not just convex in v but in (x,v), it turns out that $H(t,x,p)$ is not just convex in p but concave in x. The subgradient sets $\partial_x H(t,x,p)$ (concave sense) and $\partial_p H(t,x,p)$ (convex sense) are therefore welldefined [5]. The relation

$$(6) \qquad \dot{\bar{x}}(t) \in \partial_p H(t,\bar{x}(t),\bar{p}(t)), \ -\dot{\bar{p}}(t) \in \partial_x H(t,\bar{x}(t),\bar{p}(t))$$

is the generalized Hamilton condition. If H were differentiable as in classical mathematics, it would reduce to the equations

$$\dot{\bar{x}}(t) = \nabla_p H(t,\bar{x}(t),\bar{p}(t)), \ -\dot{\bar{p}}(t) = \nabla_x H(t,\bar{x}(t),\bar{p}(t)).$$

An absolutely continuous trajectory $\bar{x}(t)$ is said to be an extremal for L over an interval I if there is an absolutely continuous $\bar{p}(t)$ (called a co-extremal for x) such that the Hamiltonian condition (6) holds (for almost every t in I). On the other hand, $\bar{x}$ is said to be piecewise optimal for L over I if for every finite subinterval $[t_0,t_1] \subset I$ one has

$$(7) \qquad \int_{t_0}^{t_1} L(t,x(t),\dot{x}(t))dt \leq \int_{t_0}^{t_1} L(t,\bar{x}(t),\dot{\bar{x}}(t))dt$$

for all (absolutely continuous) $x(t)$ over $[t_0,t_1]$ such that $x(t_0) = \bar{x}(t_0)$, $x(t_1) = \bar{x}(t_1)$. The main result about these concepts in the present setting is the following.

THEOREM 1 [1], [2], [3]. If x is an extremal for L, then x is piecewise optimal for L. If x is piecewise optimal for L and certain "constraint qualifications" are fulfilled, then x is an extremal for L.

The exact nature of the "constraint qualifications" will not be discussed here; see [2], [3]. Basically one needs to know that the pair $(\bar{x}(t_0),\bar{x}(t_1))$ always belongs to the relative interior of the (convex) set of all pairs $(x(t_0),x(t_1))$ corresponding to trajectories for which the integral on the left of (7) is finite, and also that $\bar{x}(t)$ does not touch the boundary of the natural "state constraint set"

$$\{x \in R^n | v \in R^n \text{ with } L(t,x,v) < \infty\}.$$

(If the second condition fails, a more general theory must be invoked in which $p(t)$ is not absolutely continuous and may have jumps. The corresponding version of the Hamiltonian equation has been developed in [3]. This is indeed the situation that must be dealt with in economic applications where $x(t)$ is a nonnegative vector of goods, some components of which may well vanish from time to time.)

In economics, the variables $p(t)$ usually have an interpretation as prices of some kind. It is of great interest, therefore, that they have optimality properties relative to a function M dual to L, namely

$$M(t,p,w) = \sup_{(x,v)\in R^{2n}} \{p.v+x.w-L(t,x,v)\},$$

$$L(t,x,v) = \sup_{(p,w)\in R^{2n}} \{p.v+x.w-M(t,p,w)\}.$$

THEOREM 2 [1]. If x is an extremal for L with co-extremal p, then p is an extremal for M with co-extremal x, and hence in particular p is piecewise optimal for M.

For the case of the economic model (4), one obtains

$$(8) \qquad H(t,x,p) = \sup_{v\in R^n} \{e^{-\rho t} U(e^{-\gamma t}x, e^{-\gamma t}v)+p.v\}$$

$$= e^{-\rho t} h(e^{-\gamma t}x, e^{\delta t}p)$$

where δ is the interest rate defined by

(9) $\delta = \rho + \gamma$,

and h is the concave-convex function defined by

(10) $$h(k,q) = \sup_{z \in R^n} \{q.z+U(k,z)\}.$$

The Hamiltonian condition (7) has a rather complicated expression in terms of $\bar{x}(t)$ and $\bar{p}(t)$, but in terms of

(11) $$\bar{k}(t) = e^{-\gamma t}\bar{x}(t), \ \bar{q}(t) = e^{\delta t}\bar{p}(t),$$

it takes the autonomous form

(12) $$\begin{aligned} \dot{\bar{k}}(t) &\in \partial_q h(\bar{k}(t),\bar{q}(t)) - \gamma\bar{k}(t), \\ -\dot{\bar{q}}(t) &\in \partial_k h(\bar{k}(t),\bar{q}(t)) - \delta\bar{q}(t). \end{aligned}$$

It follows from Theorem 1 that <u>every trajectory</u> k(t) <u>satisfying</u> (12) <u>has the piecewise optimality property in</u> (2) (and the converse is "almost" true).

The function dual to L in this model is

(13) $$\begin{aligned} M(t,p,w) &= \sup_{(x,v)\in R^{2n}} \{p.v+x.w+e^{-\rho t}U(e^{-\gamma t}x,e^{-\gamma t}v)\} \\ &= e^{-\rho t}V(e^{\delta t}p,e^{\delta t}w) \end{aligned}$$

where

(14) $$V(q,s) = \sup_{(k,z)\in D} \{q.z+s.k+U(k,z)\}.$$

According to Theorem 2, the trajectories $q(t)$ appearing in (12) have the piecewise optimality property that for every finite subinterval $[t_0,t_1]$ one has

$$(15) \qquad \int_{t_0}^{t_1} e^{-\rho t} V(q(t),\dot{q}(t) - \delta q(t))dt \geq \int_{t_0}^{t_1} e^{-\rho t} V(\bar{q}(t),\dot{\bar{q}}(t) - \delta\bar{q}(t))dt$$

for all trajectories $q(t)$ over $[t_0,t_1]$ with $q(t_0) = \bar{q}(t_0)$, $q(t_1) = \bar{q}(t_1)$.

Here q can be interpreted as a vector of dated prices and $r = -s$ as a vector of rents: $\dot{q} = \delta q - r$. Thus $V(q,s)$ represents the maximum rate at which "value" can be created in the economy.

A big advantage in the study of (12) (and more generally (6)) is that this condition is an "ordinary differential equation with multivalued right side".
It is known, for example, that a solution $(k(t),q(t))$ exists over an interval $[t_0,t_0 + \varepsilon)$ starting from any point $(k(t_0),q(t_0)) = (k_0,q_0)$ interior to the region where h is finite (cf. [6], [7]). For the most part, the solutions turn out to be unique despite the multivaluedness, although branching can sometimes occur.

In the context of the infinite horizon problem (1), a critical question is how to single out, from among the trajectories $k(t)$ with $k(0) = k_0$ that satisfy (12) for some $q(t)$ (and there seems more or less to be one such for each choice of q_0), a trajectory worthy of being deemed "optimal" (or at least "extremal") over the whole interval $[0,\infty)$. No limitations are imposed a priori on the behavior of $k(t)$ as $t \to \infty$ (free endpoint problem). Heuristic considerations lead one to believe that there should "usually" be just one trajectory $k(t)$ of the desired type for each k_0 (in a reasonable region) and this seems to suggest a correspondence between k_0 and q_0 whose graph forms a sort of n-dimensional manifold in R^{2n}. The corresponding special trajectories $(k(t),q(t))$ would trace out this manifold.

If so, then in looking at examples of dynamical systems of the form (12) we should readily be able to detect a special n-dimensional manifold that is the natural candidate for expressing "optimality" over $[0,\infty)$. One approach to this question is to try to analyze behavior about a rest point (constant solution) to the system.

A rest point (k^*,q^*) of (12) is characterized by the relations

$$(16)\qquad \begin{aligned} 0 &\in \partial_q h(k^*,q^*) - \gamma k^*, \\ 0 &\in \partial_k h(k^*,q^*) - \gamma q^*. \end{aligned}$$

These are equivalent to the condition that

$$(17)\qquad 0 \in \partial_q \bar{h}(k^*,q^*),\ 0 \in \partial_k \bar{h}(k^*,q^*),$$

where

$$\bar{h}(k,q) = h(k,q) - \gamma k^* q - \delta k q^*,$$

and (17) means that (k^*,q^*) is a <u>minimax saddle point</u> of the function $\bar{h}$ (which, like h, is concave-convex). What might this imply for the behavior of Hamiltonian system (12) around (k^*,q^*)?

If h were actually twice differentiable, it would be possible to write the system in the form $(\dot{k},\dot{q}) = F(k,q)$ and analyze the behavior in terms of the matrix of derivatives of F at (k^*,q^*) in the classical manner of the theory of ordinary differential equations. If h were in fact strongly concave in k and strongly convex in q, the Jacobian of F with respect to k would be negative definite at k^*, while the Jacobian with respect to q would be positive definite. Thus the matrix in question would have n negative and n positive eigenvalues, so that system would have a <u>dynamic saddle point</u> at (k^*,q^*). This means that there would exist (locally) an n-dimensional manifold traced by the solutions $(k(t),q(t))$ that coverage to (k^*,q^*) as $t \to \infty$, as well as another n-dimensional manifold traced by the solutions that diverge from (k^*,q^*) at $t = -\infty$, the two manifolds intersecting only in the point (k^*,q^*) itself.

Karl Shell focused on this idea in his study of economic growth models and was led to conjecture that the picture of dynamic saddle point behavior should generalize somehow to the case where h is not differentiable. Moreover, the trajectories that are "optimal" over $[0,\infty)$ should be the ones converging to (k^*,q^*) as $t \to \infty$. For the economic background, see the articles [8] and [9] of Cass and Shell.

This conjecture was verified by Rockafellar in [10] for the case $\rho = 0$ $(\delta = \gamma)$ with h strictly concave-convex and in [11] for $\rho > 0$ $(\delta > \gamma)$ with h strongly concave-convex. (There is a mistake in the proof of Proposition 2' of [11] with invalidates the assertions made in the article about the complementary manifold of Hamiltonian trajectories diverging from (k^*,q^*) at $t = -\infty$ when $\rho > 0$, but this does not affect the main results, which concern the trajectories converging to (k^*,q^*).) In the case of $\rho = 0$, "optimality" must be interpreted in a certain relative sense. For $\rho > 0$, it is necessary to limit attention in (1) to trajectories $k(t)$ which do not grow at a rate faster than ρ. It must also be supposed that ρ is not too large.

The complications involved in establishing "true" optimality of some sort, and the serious restrictions on the nature of h and ρ that are entailed, bring one to the view that "optimality" over $[0,\infty)$ may not be the natural concept to be aiming at in models like (1). The justification usually given for the infinite horizon is that it enables one to avoid the selection of a particular terminal time τ and the awkward decision about what the levels of goods or prices should be at that time. However, there are other ways of avoiding this dilemma.

For example, one could consider for each time τ the trajectories $k(t)$ that would solve (1) with ∞ replaced by τ (no constraint being imposed on $k(\tau)$) and then see what trajectories these converge to as $\tau \to \infty$. Such limit trajectories would be a natural object of study. They would again be "piecewise optimal", but not necessarily optimal in any sense with respect to the integral (1) over $[0,\infty)$ (which anyway might not be well defined). There is reason to believe that this is the desired class of trajectories that exhibits the dynamic saddle point behavior (approaching a rest point as $t \to \infty$) in the many cases where the Hamiltonian system has such behavior and yet "optimality over $[0,\infty)$" cannot be established.

Convex Processes. The subject of discussion is related more closely than might be supposed to the theory of economic models in which the evolution of the state $X(t)$ (a vector of goods, resources, labor, etc.) is governed by $(X(t),\dot{X}(t)) \in T$, where T is a nonempty closed convex set in $R^N \times R^N$. In such a setting there is no real loss of generality (and considerable advantage) in taking T to be a cone and writing the dynamics in the form

$$(18) \qquad \dot{X}(t) \in A(X(t)).$$

Since the graph of the multifunction A is a closed convex cone containing the origin, A is called a closed convex process. The general theory of convex processes has been developed in [5, § 39]; for the special "monotone" and "polyhedral" cases, see [12] and [13], respectively. Convex processes play a large role in the 1973 book of Makarov and Rubinov on economic dynamics (translated 1977 by Springer-Verlag [14]).

If we associate with A the convex function

$$(19) \qquad L(X,V) = \begin{cases} 0 & \text{if } V \in A(X), \\ +\infty & \text{if } V \notin A(X), \end{cases}$$

the problem (3) appears rather degenerate. Indeed, one has

$$(20)\qquad \int_{t_0}^{t_1} L(X(t),\dot{X}(t))dt = \begin{cases} 0 & \text{if } X \text{ satisfies (18),} \\ +\infty & \text{otherwise.} \end{cases}$$

Nevertheless, the corresponding Hamiltonian system is very interesting. The Hamiltonian function is

$$(21)\qquad H(X,P) = \sup_{V \in A(X)} P.V.$$

This is not only concave in X and convex in P but positively homogeneous in each of these variables separately. For each $P \in R^N$, let

$$(22)\qquad A^*(P) = \{W | W.X \geq P.V,\ X,\ V \in A(X)\}.$$

(The multifunction A^* is the closed convex process adjoint to A.) The Hamiltonian condition

$$(23)\qquad \dot{\bar{X}}(t) \in \partial_P H(\bar{X}(t),\bar{P}(t)),\ -\dot{\bar{P}}(t) \in \partial_X H(\bar{X}(t),\bar{P}(t)),$$

is then equivalent to

$$(24)\qquad \begin{aligned} &\sup \{\bar{P}(t).V | V \in A(\bar{X}(t))\} \text{ attained at } \dot{\bar{X}}(t), \\ &\inf \{W.\dot{\bar{X}}(t) | W \in A^*(\bar{P}(t))\} \text{ attained at } -\dot{\bar{P}}(t). \end{aligned}$$

It can also be written simply as

$$(25)\qquad \begin{aligned} &\bar{X}(t) \in A(\bar{X}(t)),\ -\bar{P}(t) \in A^*(\bar{P}(t)), \\ &\bar{P}(t).\dot{\bar{X}}(t) \equiv -\dot{\bar{P}}(t).\bar{X}(t). \end{aligned}$$

Observe that the last relation is equivalent to

$$(26)\qquad \bar{X}(t).\bar{P}(t) \equiv \text{const.}$$

Trajectories $\bar{X}(t)$ which satisfy (25) for some $\bar{P}(t) \neq 0$ are said to be price-supported or competitive. It is remarkable that such trajectories and their "supports" can be generated by solving the ordinary differential "equation" (23) from arbitrary initial points (X_0,P_0) inside the region where H is finite, just as with the Hamiltonian systems discussed earlier [6], [7].

The origin (0,0) is always a rest point of (23), but it usually lies on the boundary of the region where H is finite. A more promising class of points for study is obtained through change of variables. Setting

(27) $$\bar{K}(t) = e^{-\gamma t}\bar{X}(t), \; \bar{Q}(t) = e^{\delta t}\bar{P}(t),$$

for arbitrary real numbers γ and δ, one can express the Hamiltonian condition in the form

(28) $$\begin{aligned} \dot{\bar{K}}(t) &\in \partial_P H(\bar{K}(t),\bar{Q}(t)) - \gamma\bar{K}(t), \\ -\dot{\bar{Q}}(t) &\in \partial_X H(\bar{K}(t),\bar{Q}(t)) - \delta\bar{Q}(t), \end{aligned}$$

or equivalently

(29) $$\dot{\bar{K}}(t) + \gamma\bar{K}(t) \in A(\bar{K}(t)), \; -\dot{\bar{Q}}(t) + \delta\bar{Q}(t) \in A^*(\bar{Q}(t)),$$

$$\bar{Q}(t).(\dot{\bar{K}}(t) + \gamma\bar{K}(t)) = -\dot{\bar{Q}}(t) + \delta\bar{Q}(t)).\bar{K}(t)).$$

(This transformation makes use of the homogeneity of H.) A rest point (K^*,Q^*) of the transformed system is characterized by the relations

(30) $$\gamma K^* \in A(K^*), \; \delta Q^* \in A^*(Q^*),$$

$$0 = (\delta - \gamma)K^*.Q^* = \rho K^*.Q^*$$

(where (9) is used now as the definition of ρ).

The study of the vectors K^* and Q^* satisfying (3) for various choices of γ and δ amounts to the generalized eigenvalue theory for the process A and its adjoint. In the case of A "monotone", it is closely related to the theory of growth and interest rates for the Gale-Von Neumann model (cf. [12], [13], [14]). Presumably the dynamic system (28) should exhibit a kind of "turn-pike" behavior around rest points (K^*,Q^*) in (30) for which $K^*.Q^* = 0$ (implying $\delta = \gamma$), or in other words, such that (K^*,Q^*) is a "nondegenerate" minimax saddle point for the concave-convex function

$$\phi_\lambda(K,Q) = H(K,Q) - \lambda K.Q \ (\lambda = \delta = \gamma).$$

It would be interesting to see this worked out in detail, which has not yet been done. The "turnpike" behavior should correspond to the geometric picture of a dynamic saddle point.

In fact, the theory of the inhomogeneous Hamiltonian system (12) can be recast to fit the mold of a homogeneous system associated with a closed convex process A. Consider a decomposition $R^N = R \times R^n \times R$ with corresponding notation

$$(31) \qquad X = (x_\ell,x,x_c),\ P = (p_\ell,p,p_c).$$

Let the graph of A be the closure of the set of all pairs

$$(X,V) = (x_\ell,x,x_c,v_\ell,v,v_c)$$

such that

$$(32) \qquad x > 0,\ v_c \leq x_\ell U(x/x_\ell,v/x_\ell) + \delta x_c,\ v_\ell = \gamma x_\ell.$$

The graph of the adjoint A^* is then the closure of the set of all pairs

$$(P,S) = (p_\ell,p,p_c,s_\ell,s,s_c)$$

such that

(33) $\quad p_c > 0,\ s_\ell \geq p_c\ V(p/p_c, -s/p_c) + \gamma p_\ell,\ s_c = \delta p_c,$

where V is the convex function in (14). The corresponding Hamiltonian is

(34) $\quad H(X,P) = x_\ell p_c h(x/x_\ell, p/p_c) + \gamma x_\ell p_\ell + \delta x_c p_c$

$\qquad$ for $x_\ell > 0,\ p_\ell > 0,$

where h is given by (10). (For $x_\ell = 0$ or $p_\ell = 0$, the values of H are obtained from (34) by a limit process; for $x_\ell < 0$ or $p_\ell < 0$, the values of H are infinite.)

The dynamical relation $\dot{X} \in A(X)$ reduces under (32) to

(35) $\quad x_\ell(t) = \alpha e^{\gamma t} \quad (\alpha > 0)$

$$\dot{x}_c(t) \leq \alpha e^{\gamma t} U(e^{-\gamma t} x(t)/\alpha, e^{-\gamma t}\dot{x}(t)/\alpha) + \delta x_c(t).$$

The interpretation is that x_ℓ represents a basic factor that grows at a constant rate γ (positive, negative or zero!); the parameter α merely sets the scale and can just as well be chosen as 1. The variable x_c measures "utility satisfaction" and is typically negative; it would grow (more negative) at the rate δ if this tendency were not counteracted by continual inputs of utility dependent on the vectors x/x_ℓ and $\dot{x}/x_\ell$ (quantities of goods per unit of the basic factor). Similarly, the dual dynamic relation $-\dot{P} \in A^*(P)$ reduces under (33) to

(36) $\quad p_c(t) = \beta e^{-\delta t} \quad (\beta > 0)$

$$-\dot{p}_\ell(t) \geq \beta e^{-\delta t} V(e^{\delta t} P(t)/\beta, e^{\delta t}\dot{p}(t)/\beta) + \gamma p_\ell(t).$$

Again β is just a scale parameter that can be taken as 1.

The Hamiltonian system (23) for the function (34) takes on a particularly simple form when expressed equivalently as in (28) in terms of

$$(\bar{k}_\ell(t),\bar{k}(t),\bar{k}_c(t)) = e^{-\gamma t}(\bar{x}_\ell(t),\bar{x}(t),\bar{x}_c(t)),$$

$$(\bar{q}_\ell(t),\bar{q}(t),\bar{q}_c(t)) = e^{\delta t}(\bar{p}_\ell(t),\bar{p}(t),\bar{p}_c(t)),$$

namely:

$$\bar{k}_\ell(t) \equiv \alpha,\ \bar{q}_c(t) \equiv \beta,$$

$$\dot{\bar{k}}(t) \in \alpha\partial_q h(\bar{k}(t)/\alpha,\bar{q}(t)/\beta) - \gamma\bar{k}(t)$$

(37) $$-\dot{\bar{q}}(t) \in \beta\partial_k h(\bar{k}(t)/\alpha,\bar{q}(t)/\beta) - \delta\bar{q}(t)$$

$$\dot{\bar{k}}_c(t) = \alpha U(\bar{k}(t)/\alpha,[\dot{\bar{k}}(t) + \gamma\bar{k}(t)]/\alpha) + \rho\bar{k}_c(t)$$

$$-\dot{\bar{q}}_\ell(t) = \beta V(\bar{q}(t)/\beta,[\dot{\bar{q}}(t) - \delta\bar{q}(t)]/\beta) - \rho\bar{q}_\ell(t).$$

Taking $\alpha = 1 = \beta$, one can write this as the previous system (12) for h, augmented by the equations (for all $\tau > 0$):

(38)
$$\bar{k}_c(\tau) = e^{\rho\tau}[\bar{k}_c(0) + \int_0^\tau e^{-\rho t}U(\bar{k}(t),\dot{\bar{k}}(t) + \gamma\bar{k}(t)dt],$$

$$\bar{q}_\ell(\tau) = e^{\rho\tau}[\bar{q}_\ell(0) - \int_0^\tau e^{-\rho t}V(\bar{q}(t),\dot{\bar{q}}(t) - \delta\bar{q}(t)dt].$$

This demonstrates that the inhomogeneous system (12) can indeed be treated in terms of a special case of the homogeneous system (28). The analysis of rest points carries over at the same time. As a matter of fact, for the convex process A in question, a vector pair

(39) $$K^* = (1,k^*,k_c^*),\ Q^* = (q_\ell^*,q^*,1),$$

is a rest point for (28) (i.e. satisfies (30)) if and only if (k^*,q^*) is a rest point for (12) and (from 37))

(40) $\qquad \rho k_c^* = -U(k^*,\gamma k^*), \ \rho q_\ell^* = V(q^*,-\delta q^*).$

Of course, due to the special way the numbers γ and δ enter the definition of A, they are the unique values for which (30) has a solution $K^* \neq 0$, $Q^* = 0$.

It is interesting to note that the rest points (K^*,Q^*) just described necessarily have $K^*.Q^* = 0$, however. Despite this, the analysis of the homogeneous system around (K^*,Q^*) is important, because it corresponds to the inhomogeneous system. Thus one apparently should <u>not</u>, in the general study of (28), limit attention to rest points (30) such that $K^*.Q^* \neq 0$.

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